Non-commutative Valuation Rings and Their Global Theories

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Introduction

During the last twenty years the theory of non-commutative valuation rings has been developed by many authors for different reasons. The main progress in the general theory has been made after N. I. Dubrovin introduced his new type of valuation rings which are called Dubrovin valuation rings. These rings are not only defined for division rings but also for simple Artinian rings especially for central simple algebras. As we know, there are three types of non-commutative valuation rings, which are called total valuation rings, invariant valuation rings and Dubrovin valuation rings.

Let K be a division ring. A subring V of K is called *total valuation ring* of K if for any non-zero element $a \in K$, either $a \in V$ or $a^{-1} \in V$. A total valuation ring V of a division ring K is called an *invariant valuation ring* if $aV = Va$ for all $a \in K$. An order R in a simple Artinian ring Q is called a *Dubrovin valuation ring* of Q if R is Bezout and $R/J(R)$ is simple Artinian, where $J(R)$ is the Jacobson radical of R. We see that every invariant valuation ring and every total valuation ring V is clearly a Dubrovin valuation ring. However, the converse is not necessarily true.

In this thesis, we study about non-commutative valuation rings in particular about Dubrovin valuation rings and their global theories, say Prüfer rings. Moreover, we give some examples of v-Bezout rings which are the generalization of commutative GCD-domains.

In Chapter 1, we give some elementary properties of non-commutative valuation rings, which are used in the next Chapters. We refer to [MMU] for details concerning with orders, Dubrovin valuation rings, Prüfer orders and primary ideals.

Let R be an order in a ring Q. A right R-submodule I of Q is called a *right R-ideal* of Q if (i) $I \cap U(Q) \neq \varnothing$, where $U(Q)$ is the unit group of Q and (ii) there exists $c \in U(Q)$ such that $cI \subseteq R$. A *left R-ideal* of Q is defined similarly. A right and left R-ideal is called an *R-ideal*. For a right R-ideal I of Q, we set $O_r(I) = \{ q \in Q \mid Iq \subseteq I \}$, *the right order of I* and $O_l(I) = \{ q \in Q \mid qI \subseteq I \}$, *the left order of I*. An element c in Q is called a *right stabilizing element* of R if cR is an R-ideal and we denote by r-st(R) = $\{ c \in Q \mid cR$ is right stabilizing$\}$. We say that c is *stabilizing* if $cR = Rc$ and denote by st(R) = $\{ c \in Q \mid c$ is stabilizing$\}$. For any ideal I of a ring R, we denote by $\sqrt{I}$ = $\cap\{ P \mid P \in$ Spec(R) with $P \supseteq I\}$ the *prime radical* of I which is a semi-prime ideal. An ideal A of R is called *right* $\sqrt{I}$ *–primary* if $aRb \subseteq A$, where $a, b \in R$, implies either $a \in A$ or $b \in \sqrt{I}$. Similarly, left primary ideals are defined. In [BMU], they have described all right primary ideals of R.

In Chapter 2, we investigate the structure of all R-ideals by usage of stabilizing elements and primary ideals by using some results from [BMU]. If I is an R-ideal and I is not finitely generated as a right R-ideal such that $O_r(I) = S = O_l(I)$ and suppose that $J(S)$ is Archimedean, it is proved that $I = cA$ for some $c \in$ st(S) and A, a right and left $J(S)$-primary ideal (see Theorem 2.2.3). In the case Q is finite dimensional over its center, we obtain: (1) If I is finitely generated as a right R-ideal, then $I = cR = Rc$ for some $c \in$ st(R), (2) If I is not finitely generated as a right R-ideal such that $J(S)$ is Archimedean, then $I = cA = Ac$ for some $c \in$ st(S) and A, a right and left $J(S)$-primary ideal, (3) If I is not finitely generated as a right R-ideal such that $J(S)$ is limit prime, then I is one of the following three; $I = cS = Sc$ for some $c \in$ st(S), $I = cJ(R) = J(R)c$ for some $c \in$ st(R) and $I = \bigcap c_\lambda R_\lambda$ for some $c_\lambda \in$ st(R_λ), where $R_\lambda = R_{P_\lambda}$ and P_λ runs over all Archimedean prime ideals with $P_\lambda \subset J(S)$ (see Proposition 2.2.4). Furthermore, a counter example is given to show that Proposition 2.2.2 (2)(a) is not necessarily held if Q is infinite dimensional over its center.

A ring is called *right (left) bounded* if any essential right (left) ideal contains a non-zero (two-sided) ideal. A ring is just called *bounded* if it is both right bounded and left bounded. Let S be a ring. We say that S is *fully bounded* if S/P is bounded for any prime ideal P of S. Let R be a Dubrovin valuation ring in a simple Artinian ring Q and let $P \in$ G-Spec(R), the set of all Goldie prime ideals of R, with $P \neq J(R)$ and set P_1 =

$\cap\{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $P_\lambda \supset P\}$. Then, in [BMO,(6)], they have shown that the following four cases only occur:

(1) P is lower limit, i.e., $P = P_1$. Otherwise, $P_1 \supset P$ is a prime segment.

(2) $P_1 \supset P$ is Archimedean.

(3) $P_1 \supset P$ is simple.

(4) $P_1 \supset P$ is exceptional, i.e., there exists a non-Goldie prime ideal C such that $P_1 \supset C \supset P$.

In Chapter 3, we investigate those results under an additional condition that R is fully bounded. It is shown that for a Dubrovin valuation ring R of a simple Artinian ring Q, R is fully bounded iff (1) and (2) only hold.

Moreover, for any regular element c in $J(R)$, we define $P(c) = \cap\{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $c \in P_\lambda\}$, a Goldie prime ideal. R is called *locally invariant* if $cP(c) = P(c)c$ for any regular element c in $J(R)$. Let R be a Dubrovin valuation ring of a simple Artinian ring Q. It is shown that R is fully bounded if and only if it is locally invariant.

If Q is of finite dimensional over its center, then R is always fully bounded. In the end of this chapter, we give several examples of fully bounded Dubrovin valuation rings of Q with infinite dimension over its center.

In Chapter 4, we study non-commutative GCD-domains. An Ore domain S is called *right (left) v-Bezout* if I_v is a principal for any finitely generated right ideal I of S. S is said to be *v-Bezout* if it is right v-Bezout as well as left v-Bezout. This ring is a non-commutative version of a commutative GCD-domain. In the commutative rings, [Gi] proved if R is a GCD domain, so is $R[x]$. Inspired by [Gi], we prove if V is a total valuation ring of a division ring K, then $R = V[x^r, \sigma \mid r \in \mathrm{Q}_0]$ is v-Bezout where Q_0 the set of non-negative rational numbers, σ: $\mathrm{Q}_0 \rightarrow \mathrm{Aut}(V)$ is defined by $\sigma(r + s) =$

$\sigma(r).\sigma(s)$ for any r, $s \in \mathrm{Q}_0$, and the multiplication in R is defined by $x^r a = \sigma(r)(a)$ x^r for any $a \in V$ and $r \in \mathrm{Q}_0$.

In Chapter 5, we study prime ideals of any overring of a non-commutative PI Prüfer ring. We define $I^{-1} = \{ q \in Q \mid IqI \subseteq I \}$), the *inverse* of I. Following [AD], R is called *right Prüfer* if for every finitely generated right R-ideal I, $I^{-1}I = R$, $I I^{-1} = O_l(I)$. *Left Prüfer rings* are defined similarly. In [D_2], he proved that any prime ideal of a PI Prüfer ring is localizable. In the case when prime rings satisfying a PI, [Mo] studied PI Prüfer rings under some conditions. By using some results in [D_2] and [Mo], we shall prove if S is an overring of a prime Goldie ring R and suppose that R is Prüfer satisfying a polynomial identity, then Spec(S) = $\{PS \mid P \in \text{Spec}(R) \text{ with } PS \subset S\}$ and $S = \cap R_P$, where P runs over all $P \in$ Spec(R) with $PS \subset S$.

CHAPTER 1

Some elementary properties

In this chapter, we give some elementary properties of orders, non-commutative Dubrovin valuation rings and Prüfer orders. We refer to [MMU] and [MR] for details concerning with orders.

1.1. Some elementary properties of orders

In this section, we give some definitions, notations and elementary properties of orders. For a ring R, we denote by $U(R)$ the set of all units of R and by $C_R(0)$ the set of all regular elements (that is, non-zero divisors) of R.

Let C be a multiplicatively closed subset of a ring R. We say that R *satisfies the right Ore condition with respect to* C or that C is called a *right Ore set of* R if, for any $a \in R$ and $c \in C$, there exist $b \in R$ and $d \in C$ such that $ad = cb$. If $C \subseteq C_R(0)$, then it is called a *regular right Ore set of* R. Similarly, we can define a *(regular) left Ore set* of R. If C is a (regular) right and left Ore set of R, then it is simply called *(regular) Ore set* of R.

Let C be a regular right Ore set of a ring R. An overring T of R is called the *right quotient ring of* R *with respect to* C if

(i) $c \in U(T)$ for any $c \in C$ and

(ii) for any $q \in T$, there exist $a \in R$ and $c \in C$ such that $q = ac^{-1}$.

We denote the ring T by R_C. We note that for a multiplicative subset C of R with $C \subseteq C_R(0)$, the right quotient ring R_C of R with respect to C exists if and only if C is a regular right Ore set of R ([MR, Chap. 2]).

A subring R of a ring Q is called a *right order in* Q if Q is the right quotient ring of R with respect to $C_R(0)$, and sometimes we denote the ring Q by $Q(R)$. In particular, R is a right order in Q if and only if $C_R(0)$ is a right Ore set of R. Similarly, we can define a *left order in* Q and a ring which is both a right and left order in Q is called an *order in* Q.

Let R be a ring and let M be a right R-module. An R-submodule L of M is called *essential* if $L \cap N \neq 0$ for any non-zero R-submodule N of M. By Zorn's lemma, we note that for any R-submodule L of M, there exists an R-submodule L' of M such that $L \cap L' = 0$ and $L \oplus L'$ is essential in M ([MR, (2.2.2(v))]). If a right ideal I of R is an essential R-submodule of R, then I is called an *essential right ideal*. A right R-module U is said to be *uniform* if, for any non-zero R-submodule U_1 and U_2 of U, $U_1 \cap U_2 \neq 0$, that is, any non-zero R-submodule of U is an essential R-submodule of U.

A right R-module M is said to have *finite Goldie dimension* if it contains no infinite direct sum of non-zero R-submodules. For any subset X of R, we set $r_R(X) = \{ a \in R \mid Xa = 0\}$ and call it the *right annihilator of* X. Sometimes we denote $r_R(X)$ by $r(X)$. The *left annihilator of* X is defined similarly. A ring R is called a *right Goldie ring* if R satisfies the ascending chain condition (acc) on right annihilators and has finite Goldie dimension as a right R-module. A *left Goldie ring* is defined similarly and R is called a *Goldie ring* if R is a right and left Goldie ring. We have the following property ([MMU, (1.1)]).

Theorem 1.1.1. *Let R be a ring. Then the following is equivalent:*
(i) R is a (semi)-prime right Goldie ring.
(ii) R has a right quotient ring Q which is (semi)-simple Artinian, that is, R is a right order in a (semi)-simple Artinian Q.

Let R be an order in a ring Q. A right R-submodule I of Q is called a *right R-ideal of Q* if $I \cap U(Q) \neq \varnothing$ and there exists $c \in U(Q)$ such that $cI \subseteq R$. A right R-ideal I of Q is said to be *integral* if $I \subseteq R$. Similarly, we can define a *left R-ideal of Q*. A right and left R-ideal is called an R-ideal.

Let R an order in a ring Q. For any subsets X and Y of Q, we set $(X : Y)_l = \{ q \in Q \mid qY \subseteq X$, $(X : Y)_r = \{ q \in Q \mid Yq \subseteq X \}$ and $X^{-1} = \{ q \in Q \mid XqX \subseteq X \}$. For a right R-ideal I of Q, we set $O_r(I) = (I : I)_r = \{ q \in Q \mid Iq \subseteq I \}$ and we called it *the right order of I*. The *left order of I* is defined similarly. A right and left R-ideal is called an *R-ideal*. Then we have the following lemma ([MMU, 1.2]).

Lemma 1.1.2. *If R is an order in a ring Q and I is a right R-ideal of Q, then*

(1) *$O_r(I)$ and $O_l(I)$ are orders in Q,*

(2) *I is a left $O_l(I)$–ideal and a right $O_r(I)$–ideal, and*

(3) *$(R:I)_l$ is a left R-ideal and a right $O_l(I)$–ideal.*

1.2. Some elementary properties of Dubrovin valuation rings

In this section, we give some elementary characterizations of Dubrovin valuation rings and its ideal theory.

Let D be a division ring, $(G, +)$ be a totally ordered group and let $U(D)$ be the set of all units in D. A surjective mapping v: $U(D) \to G$ is called a *valuation* on D if it is satisfying ([Sc]):

(1) For any $a, b \in D$, $v(ab) = v(a) + v(b)$.

(2) $v(a + b) \geq \min \{ v(a), v(b) \}$ if $b \neq -a$.

If v is a valuation on D, then $V = \{ a \in U(D) \mid v(a) \geq 0 \} \cup \{0\}$. Then V is an invariant valuation ring. In this case G is called *value group* of V.

A ring R is called *right Bezout* if any finitely generated right ideal of R is principal. The *left Bezout* is defined similarly. A ring R is called *Bezout* if it is right and left Bezout.

Let R be a subring of a ring S. R is called a *right n-chain ring in S* if for any n+1 elements a_0, a_1, ..., a_n in S, there is an i such that $a_i \in \sum_{k \neq i} a_k R$. A right n-chain ring in itself is called a *right n-chain ring*. A *left n-chain ring* is defined similarly. An *n-chain ring* is a right and left n-chain ring. Then we have the following properties ([MMU, (5.8), (5.11) and (5.12)]).

Lemma 1.2.1. *Let R be a semi-simple ring, that is, $J(R) = 0$. Then R is Artinian if and only if R is a right n-chain ring for some n.*

Theorem 1.2.2. *Let R be a subring of a simple Artinian ring Q. Then the following conditions are equivalent:*

(1) *R is a Dubrovin valuation ring of Q.*

(2) *R is a local semi-hereditary order in Q.*

(3) *R is a local Bezout order in Q.*

(4) *R is a local n-chain ring in Q for some n with $d(\overline{R}) \geq n$, where $\overline{R} = R/J(R)$.*

Corollary 1.2.3. *Let R be a Dubrovin valuation ring of Q and let P be a prime ideal of R. If R/P is a prime Goldie ring, then R/P is also a Dubrovin valuation ring of its classical quotient ring.*

Lemma 1.2.4. ([MMU, (6.3)]). *Let R be a Dubrovin valuation ring of Q and let $T_2 \subseteq T_1$ be right R-submodules of Q such that* (1) *T_1 is regular and* (2) *there exists a subring S of $O_l(T_2) = \{ q \in Q \mid qT_2 \subseteq T_2 \}$ such that for any regular elements t_1 and $t_2 \in T_1$ there is a regular element $t \in T_1$ with $St_1 + St_2 \subseteq St$. Then either $T_1 = T_2$* or *$t_1 J(R) \supseteq T_2$ for some regular element $t_1 \in T_1$.*

By using Lemma 1.2.4, we have the following Proposition ([MMU, (6.4)])

Proposition 1.2.5. *Let R be a Dubrovin valuation ring of Q and let S be a Bezout order in Q. Then the set of regular left S- and right R-submodules of Q is linearly ordered by inclusion. In particular, the set of all R-ideals of Q is linearly ordered by inclusion.*

Let P be a prime ideal of a ring S. If $C(P) = \{ c \in S \mid c :$ regular mod $P\}$ is a regular Ore set of S then the quotient ring $S_{C(P)}$ of S with respect to $C(P)$ is denoted by S_P and is called the *localization* of S at P. Let R be a Dubrovin valuation ring of a simple Artinian ring Q and let S be an overring of R. Then $J(S) \subseteq J(R)$ and S is local ([MMU, (5.3)]). Combining Lemma 1.2.1 and Theorem 1.2.2, we have ([MMU, (6.6)])

Theorem 1.2.6. *Let R be a Dubrovin valuation ring of Q and let S be an overring of R.*

(1) $\tilde{R} = R/J(S)$ *is a Dubrovin valuation ring of* $\bar{S} = S/J(S)$.

(2) S *is a Dubrovin valuation ring of Q and $J(S)$ is a prime ideal of R.*

(3) $C_R(J(S))$ *is a regular Ore set of R and* $S = R_{J(S)}$.

The converse of Theorem 1.2.6 (1) also holds ([MMU, (6.16)]) as following.

Proposition 1.2.7. *Let S be a Dubrovin valuation ring of Q and let $\tilde{R}$ be a Dubrovin valuation ring of $\bar{S} = S/J(S)$. Then the set $R = \{ r \in S \mid [r + J(S)] \in \tilde{R} \}$ is a Dubrovin valuation ring of Q.*

Let R be a Dubrovin valuation ring of Q. Then $O_r(J(R)) = O_l(J(R)) = R$ by [MMU, (6.8)], which implies the following Lemmas ([MMU, (6.9) and (6.10)]).

Lemma 1.2.8. *Let R be a Dubrovin valuation ring of Q, A be an R-ideal of Q and $S = O_r(A)$. Then the following are equivalent:*

(1) *A is principal as a right S-ideal.*

(2) $A^{-1}A = S$.

(3) $A \supset AJ(S)$.

Lemma 1.2.9. *Let R be a Dubrovin valuation ring of Q and let A be an R-ideal of Q. Then $O_r(A) = O_l(A^{-1})$ and $O_l(A) = O_r(A^{-1})$.*

Combining Lemma 1.2.4 and Lemma 1.2.9, we have

Proposition 1.2.10 ([MMU, (6.13)]). *Let R be a Dubrovin valuation ring of Q and let A be an R-ideal of Q. Set $S = O_r(A)$ and $T = O_l(A)$.*

(1) $A_v = (S : (S : A)_l)_r = A^* = {}^*A = (T : (T : A)_r)_l$ *and* $A^* = A^{-1\,-1}$.

(2) $A^{**} = A^*$ *and* $(A^{-1})^* = A^{-1}$.

(3) *If A is not a principal right S-ideal, then $A^{-1}A = J(S)$ and $J(S)$ is not a principal right S-ideal.*

(4) *If $A \subset A^*$*, then $A^* = cS$ *and* $A = cJ(S)$ *for some regular element $c \in A^*$. In particular, $A = A^* J(S)$.*

Let I be a right R-ideal and let $S = O_r(I)$. We define $I^* = \cap\, cS$, where c runs over all elements in Q with $cS \supseteq I$. Similarly, for any left R-ideal L with $T = O_l(L)$, we define $^*L = \cap\, Tc$, where c runs over all elements in Q with $Tc \supseteq L$. The following proposition is established by a standard method ([MMU, (6.11)]).

Proposition 1.2.11. *Let R be a Dubrovin valuation ring of Q and let I be a right R-ideal of Q.*

(1) $I \subseteq I^*$.

(2) $(I^*)^* = I^*$.

(3) $(cI)^* = cI^*$ *for any* $c \in U(Q)$.

(4) $(cI)^{-1} = I^{-1}c^{-1}$ *for any* $c \in U(Q)$.

Let R be a Dubrovin valuation ring of Q. Then by [MMU, (6.8)], $O_r(J(R)) = O_l(J(R)) = R$. If $J(R)$ is not principal as $O_r(J(R))$-ideal, then $J(R)^* = R = J(R)^{-1}$ by [MMU, (6.12)].

A prime ideal P of a ring R is called Goldie if R/P is a Goldie ring. By [MMU, (6.8)], if R is a Dubrovin valuation ring of a simple Artinian ring Q and S is an overring of R then $J(S)$ is a Goldie prime ideal of R, which is localizable and R_P is a Dubrovin valuation ring with $J(R_P) \cap R = P$ ([MMU, (14.5)]). We denote by $\mathcal{B}(R)$ the set of all overrings of a ring R, Spec(R) the set of all prime ideals of R, and G-Spec(R) the set of all Goldie prime ideals of R. By [MMU, (6.7) and (14.5)], we have the following correspondence

Proposition 1.2.12. *Let R be a Dubrovin valuation ring of a simple Artinian ring Q. Then there exists a one-to-one correspondence between $\mathcal{B}(R)$ and* G-Spec(R).

Let R be a Dubrovin valuation ring of a simple Artinian ring Q. Then the intersection of Goldie prime ideals of R is also a Goldie prime ideal ([BMO, (1)])

Proposition 1.2.13. *Let R be a Dubrovin valuation ring of a simple Artinian ring Q and let $P_i \in$* G-Spec(R). *Then $P = \cap P_i$ is also in* G-Spec(R).

Let R be be a Dubrovin valuation ring of a division ring K. The following Lemma ([MMU, (8.13)]) gives a criterion of R to be a total valuation ring.

Lemma 1.2.14. *Let R be be a Dubrovin valuation ring of a division ring K. Then R is total if and only if $\overline{R} = R/J(R)$ is a division ring.*

1.3. Some elementary properties of Prüfer orders

In this section, we give some properties of Prüfer orders. Let Q be a semi-simple Artinian ring and let R be an order in Q, that is, R is a semi-prime Goldie ring. R is called a *right (left) Prüfer order* in Q if any finitely generated right (left) R-ideal is a progenerator of Mod-R (R-Mod), that is, projective and a generator of Mod-R (R-Mod). A *Prüfer order* is a right and left Prüfer order. By [MMU, (2.5)], a right Prüfer order in a semi-simple Artinian ring is left Prüfer order.

Let R be a Prüfer order in a semi-simple Artinian ring Q and let S be an overring of R, that is, $R \subseteq S \subseteq Q$. It is clear that S is an order in Q. It is proved in ([MMU, (2.6)]) that S is also Prüfer.

Proposition 1.3.1. *An overring of a Prüfer order in a semi-simple Artinian ring is also a Prüfer order.*

Let A be an ideal of a Prüfer order R in a simple Artinian ring Q. Then any element of $C(A) = \{ c \in R \mid c$ is regular modulo A$\}$ is regular ([MMU, (22.6)]). In the case A is maximal such that R/A is a semi-simple Artinian ring, then $C(A)$ is an Ore set of R and R_A is a Dubrovin valuation ring of Q ([MMU, (22.7)]).

Proposition 1.3.2. *Let A be an ideal of a Prüfer order in a simple Artinian ring Q. Then $C(A)$ consists of regular elements of R.*

Theorem 1.3.3. *Let R be a Prüfer order in a simple Artinian ring Q and let A be an ideal of R such that R/A is a semi-simple Artinian ring.*

(1) *$C(A)$ is a regular Ore set of R.*

(2) *If A is a maximal ideal of R, then R_A is a Dubrovin valuation ring of Q.*

Dubrovin has proved the following property of Prüfer order ([D_2, (4)])

Proposition 1.3.4. *Let R be a Prüfer order in a simple Artinian ring Q and let S be a Dubrovin valuation ring of Q containing R. Then $P = J(S) \cap R$ is a prime ideal of R such that $C(P)$ is a regular Ore set of R and $S = R_P$.*

The following Proposition is proved by [Mo, (3.1)]

Proposition 1.3.5. *Suppose S is a Dubrovin valuation ring and $\mathcal{R}$ is an order in $\overline{S} = S/J(S)$. Then $R = \{ r \in S \mid r + J(S) \in \mathcal{R}\}$ is Prüfer if and only if $\mathcal{R}$ is Prüfer.*

CHAPTER 2

On R-ideals of a Dubrovin valuation ring R

Throughout this Chapter, we denote by R a Dubrovin valuation ring in a simple Artinian ring Q. We use "$\subset$" or "$\supset$" for proper inclusion and "$\subseteq$" or "$\supseteq$" for inclusion. For any subset X and Y of Q, we set $(X : Y)_l = \{ q \in Q \mid qY \subseteq X \}$ and $(X : Y)_r = \{ q \in Q \mid Yq \subseteq X \}$. For an R-ideal I, we set $I_v = (R :(R :I)_l)_r$ and ${}_vI = (R :(R :I)_r)_l$. I is called a *v-ideal* if $I_v = I = {}_vI$.

In Section 2.1, we give some structures of v-ideals related to the properties of Jacobson radical.

In Section 2.2, it is described the structures of all R- ideals by usage of stabilizing elements and primary ideals.

We refer to [MMU] and [BMU] for details concerning with Dubrovin valuation rings and primary ideals.

2.1. Structure of v-ideals

For an R-ideal I, we set $O_r(I) = \{ q \in Q \mid Iq \subseteq I \}$, *the right order of* I and $O_l(I) = \{ q \in Q \mid qI \subseteq I \}$, *the left order of* I. Then we have the following

Lemma 2.1.1. *Let S be a proper overring of R. Then*

(1) $(R : S)_l = J(S)$

(2) $(R : J(S))_l = S$

Proof. (1) It is clear that $(R : S)_l \supseteq J(S)$. If $(R : S)_l \supset J(S)$, then $(R : S)_l = (R : S)_l S = S$ because $(R : S)_l$ is an ideal of R and $S = R_{J(S)}$, a contradiction.

(2) It is clear that $(R : J(S))_l \supseteq S$. To show the converse inclusion, first assume that $J(S) = sS = Ss$ for some $s \in J(S)$. Then we have $(R : J(S))_l = (R : S)_l\, s^{-1} = J(S)\, s^{-1} = S$ by (1). Next, assume that $J(S)$ is not finitely generated as a one-sided S-ideal. Then we

have $J(S) = J(S)^2$ and $O_l(J(S)) = S$ by [MMU, (6.8)] and Lemma 1.2.8, and it follows that $(R : J(S))_l \subseteq (S : J(S))_l = O_l(J(S)) = S$. Hence $(R : J(S))_l = S$.

A prime ideal P of R is said to be *Goldie prime* if R/P is a Goldie ring. By Theorem 1.2.6 and Proposition 1.2.12, P is Goldie prime if and only if R_P exists and is a Dubrovin valuation ring. We note that $J(S)$ is always Goldie prime for any overring S of R.

Lemma 2.1.2. *Let I be an R-ideal and set $S = O_r(I)$ and $T = O_l(I)$. Then*

(1) *If I is finitely generated as a right R-ideal, then $(R : I)_l I = R$.*

(2) *If I is not finitely generated as a right R-ideal, then $(R : I)_l I = J(S)$. In Particular, $(R : I)_l I$ is Goldie prime.*

Proof. (1) It is clear.
(2). If $I = aS$ for some $a \in I$, then $S \neq R$ by assumption, and so $(R : I)_l = (R : aS)_l = (R : S)_l a^{-1} = J(S) a^{-1}$ by Lemma 2.1.1(1). Hence we have $(R : I)_l I = J(S) a^{-1} aS = J(S)$. If I is not finitely generated as a right S-module, then $I = I J(S)$ by Lemma 1.2.8. It follows by Lemma 2.1.1(2) that $(R : I)_l = (R : I J(S))_l = ((R : J(S))_l : I)_l = (S : I)_l = I^{-1}$ (:= $\{ x \in Q \mid IxI \subseteq I \}$). Thus By Proposition 1.2.10 (3), $J(S) = I^{-1} I = (R : I)_l I$.

An element c in Q is called a *right stabilizing element* of R if cR is an R-ideal and we denote by r-st(R) = $\{ c \in Q \mid cR$ is right stabilizing$\}$. We say that c is *stabilizing* is $cR = Rc$ and denote by st(R) = $\{ c \in Q \mid c$ is stabilizing$\}$.

If S is a Noetherian Prüfer order in a simple Artinian ring, i.e., a Dedekind ring, then any ideal is always a v-ideal, because it is projective. However, in non-Noetherian case, this is not necessarily to be held. In the case of Dubrovin valuations rings, this depends on the properties of Jacobson radical, as it will be seen in the following proposition which is used in section 2.2.

Proposition 2.1.3.

(1) *If* $J(R) = xR = Rx$ *for some* $x \in R$, *then any* R*-ideal is a* v*-ideal*

(2) *If* $J(R) = J(R)^2$, *then* $\{ cJ(R) \mid c\in$ *r-st*$(R)\}$ *is the set of all non* v*-ideals.*

Proof. Let I be an R-ideal with $I \subset I_v$. Then $I \subseteq aJ(R) \subseteq I_v$ for some regular element $a\in I_v$ by Lemma 1.2.4. So $I_v \subseteq (aJ(R))_v = (axR)_v = axR \subseteq I_{vv} = I_v$. Thus $I_v = axR \subseteq aR \subseteq I_v$, and we have $J(R) = xR = R$, a contradiction. Hence $I = I_v$, and similarly we have $I = {}_vI$.

(2) By [MMU, (6.8), (6.12)] and Proposition 1.2.11, we have $(cJ(R))_v = c(J(R))_v = cR \supset cJ(R)$ for any $c\in$ r-st(R), and so $cJ(R)$ is not a v-ideal. Conversely, let I be an R-ideal with $I \subset I_v$. Then, by Lemma 1.2.4, $I \subseteq cJ(R) \subseteq I_v$ for some regular element $c\in I_v$. So $I_v = (cJ(R))_v = cR$. Thus $c\in$ r-st(R). Now we shall show that $I = cJ(R)$. To prove this assume, on the contrary, that $I \subset cJ(R)$. Then there is a regular element $d\in cJ(R)$ with $I \subseteq dJ(R) \subseteq I_v$. Thus, again we have $I_v = dR$, which implies $d\in$ r-st(R). On the other hand, since $d\in cJ(R)$, we have $dR \subset cJ(R)$, because dR is a v-ideal and $cJ(R)$ is not a v-ideal. Thus $I_v = dR \subset cJ(R) \subseteq I_v$, a contradiction. Hence $I = cJ(R)$.

Remark. In the case Q is finite dimensional over its center, [D_1] has obtained the following ([MMU, (7.12) and (7.5)]):

(1) $O_r(I) = O_l(I)$ for any R-ideal I.

(2) If $cR \supseteq Rc$ for some $c\in Q$, then $cR = Rc$. In particular, r-st(R) = st(R) = l-st(R) =

$\{ c \in Q \mid cR$ is left stabilizing $\}$.

(3) If $cR = Rc$, then $cS = Sc$ for an overring S of R.

However, if Q is infinite dimensional over its center, then (1) − (3) are not necessarily to be held. For example, let (K,W) and (K,V) be valued fields as in [XKM, (2.5)], namely, $W \supset V$, $\sigma \in$ Aut (K) such that $\sigma(V) = V$ and $\sigma(W) \subset W$. Set $S = W_{(1)} =$

$W + xT$ and $R = V_{(1)} = V + xT$, where $T = K[x, \sigma]_{(x)}$, the localization of $K[x, \sigma]$ at maximal ideal $(x) = xK[x, \sigma]$. S and R are both Dubrovin valuation rings, in fact, they are total valuation rings. First we note that $xSx^{-1} = \sigma(S) = \sigma(W) + \sigma(xT) \subseteq W + xT = S$. By [XKM, (1.5)], $\sigma(S) \subset S$, so that $xSx^{-1} \subset S$. Hence $I = Sx$ is an ideal of S with $I \supset xS$. Similarly, $xR = Rx$, because $\sigma(V) = V$. Furthermore, it is easily seen that $S = O_l(I) \subset x^{-1}Sx = O_r(I)$. Hence (1)-(3) are not necessarily true. In particular, $x \in$ l- st(S) but $x \notin$ st(S).

2.2. *R*-ideals of a Dubrovin valuation ring *R*

For any ideal I of a Dubrovin valuation ring R of Q, we write Spec(R) for the set of all prime ideals of R and denote $\sqrt{I} = \cap\{P \mid P \in \text{Spec}(R) \text{ with } P \supseteq I\}$ the *prime radical* of I which is a prime ideal ([MMU, (13.1)]). An ideal A of R is called *right* $\sqrt{A}$*-primary* if $aRb \subseteq A$, where $a, b \in R$, implies either $a \in A$ or $b \in \sqrt{A}$. Similarly, left primary ideals are defined. In [BMU], they have described all right primary ideals of R. So it is natural to ask the question: Describe the structure of all R-ideals by usage of stabilizing elements and primary ideals. In this section, we give a partial answer to this question in general case and a complete answer in the case Q is finite dimensional over its center after a few preliminary lemmas.

Lemma 2.2.1. *Let R be a Dubrovin valuation ring and let I be an R-ideal which is not finitely generated as a right S-ideal, where* $S = O_r(I)$. Then $(S : I)_l = (R : I)_l$.

Proof. First note that $J(S) = J(S)^2$ and so $I = I\,J(S)$, by Lemma 1.2.8 and Proposition 1.2.10(3). Hence we have $(R : I)_l = (R : IJ(S))_l = ((R : J(S))_l : I)_l = (O_l(J(S) : I)_l = (S : I)_l$ by [MMU, (6.8)], because $(R : J(S))_l = O_l(J(S))$.

Remark. In Lemma 2.2.1, we can not drop the condition that I is not finitely generated as a right S-module and $J(S) = J(S)^2$.

(1) If $S \neq R$ and $I = aS$ (see the example in the Remark of Sec. 2.1), then $(R:I)_l \subseteq Ra^{-1} \subseteq Sa^{-1} = (S:I)_l$.

(2) If $J(S) \supset J(S)^2$, then $J(S) = aS$, and so $(R:I)_l \subseteq Ra^{-1} \subseteq Sa^{-1} = (S:I)_l$, where $I = aS$.

A Goldie prime ideal P is *Archimedean* if there is a prime segment $P \supset P_0$ which is Archimedean, that is, for any $a \in P \setminus P_0$, there is an ideal $I \subseteq P$ with $a \in I$ and $P_0 = \cap I^n$ (see [BMO] and [BMU] for details concerning prime segments). Then we have the following.

Lemma 2.2.2.

(1) *Suppose that $J(R)$ is Archimedean and $J(R) = RxR$. Then $J(R)$ is principal.*

(2) *Let I be an R-ideal with $S = O_r(I)$ and $T = O_l(I)$. Suppose that $I = RqR$ and $J(S)$ is Archimedean. Then $I = aS = Ta$ for some $a \in I$.*

Proof. (1) Let $J(R) \supset P_0$ be the Archimedean prime segment. But $\mathcal{F} = \{A \mid A$ is an ideal of R and $A \not\ni x\}$. Then $\mathcal{F}$ is a non-empty inductive set, and so it contains a maximal element B. Since there are no ideals between $J(R)$ and B properly, B is prime if $J(R) = J(R)^2$. In this case, we have $B = P_0$, which contradicts the Archimedeaness. So $J(R) \supset J(R)^2$ and hence $J(R)$ is principal.

(2) To show $I = aS$ for some $a \in I$, it suffices to prove that $IJ(S) \subset I$. Suppose that on the contrary, $I = IJ(S)$. Then $q = r_1 q x_1 + \dots + r_n q x_n$ and $Sx = Sx_1 + \dots + Sx_n$, where $r_i \in R$ and $x, x_i \in J(S)$, $i = 1, \dots, n$. Now, we have $I = ISxS$. If $J(S) = SxS$, then by (1), $J(S) = sS = Ss$ and so $I = Is$. It follows that $s^{-1} \in O_r(I) = S$, a contradiction. Hence $J(S) \supset SxS$ and there is some $t \in J(S)$ with $J(S)t \supset SxS$ by Lemma 1.2.4. Then $I = ISxS \subseteq IJ(S)t \subseteq It$ and so $t^{-1} \in O_r(I) = S$, a contradiction. Hence $I \supset IJ(S)$. Thus $I = aS$ for some $a \in I$ and $I = aSa^{-1}a = O_r(I)a = Ta$ follows.

Theorem 2.2.3. *Let R be a Dubrovin valuation ring in simple Artinian ring Q. Let I be an R-ideal such that $O_r(I) = S = O_l(I)$ and I is not finitely generated as a right R-ideal. Suppose that $J(S)$ is Archimedean. Then $I = cA$ for some $c \in$* st*(S) and A a right and left $J(S)$-primary ideal.*

Proof. Let $J(S) \supset P$ be an Archimedean prime segment. By Lemma 2.1.2 (2), we have $(R:I)_l I = J(S)$. Let $\mathcal{F} = \{x \in (R:I)_l \mid \sqrt{SxSI} = J(S)\}$. Then $\mathcal{F} \neq \varnothing$, because $J(S)$ is Archimedean. First we claim that $A = SxSI$ ($x \in \mathcal{F}$) is right and left $J(S)$–primary. Since $\sqrt{A} = J(S)$, it suffices to prove that $O_r(A) = S = O_l(A)$ by [BMU, (2.5)]. It is clear that $O_r(A) \supseteq S$ and $O_l(A) \supseteq S$. If $O_r(A) \supset S$, then $O_r(A) \supseteq R_P$ because there are no Goldie prime ideals between $J(S)$ and P, and so we have $A = B R_P = R_P$, a contradiction. Similarly, $O_l(A) = S$.

Next we show that $\bigcup\{ SxS \mid x \in \mathcal{F} \} = (R:I)_l$. Since $O_r(I) = S = O_l(I)$, $(R:I)_l$ is a right S-ideal. To show that $(R:I)_l$ is a left S-ideal, first suppose that $I = aS$ for some $a \in I$. Then, by Lemma 2.1.1, $(R:I)_l = J(S)\, a^{-1}$ so that it is a left S-ideal. Suppose that I is not finitely generated as a right S-ideal. Then, by Lemma 2.2.1, $(R:I)_l$ is a left S-ideal. Hence $SxS \subseteq (R:I)_l$ for any $x \in \mathcal{F}$. Suppose that $y \in (R:I)_l$ but $y \notin \mathcal{F}$. Then we have $SySI \subseteq P$ and so $SyS \subseteq SxS$ for any $x \in \mathcal{F}$. This is a contradiction and hence $\bigcup\{ SxS \mid x \in \mathcal{F} \} = (R:I)_l$ holds.

Finally we claim that $O_r(SxS) = S$ for some $x \in \mathcal{F}$. Suppose that $O_r(SxS) \supset S$ for all $x \in \mathcal{F}$. Then $O_r(SxS) \supseteq R_P$ and so $(R:I)_l\, R_P = (R:I)_l$.

Case 1. $I = aS$ for some $a \in I$. Then $S = O_r(I) = aS\, a^{-1}$ and $J(S) = aJ(S)\, a^{-1}$ follows. By Lemma 2.1.1 (1), $(R:I)_l = (R:aS)_l = (R:S)_l a^{-1} = J(S)\, a^{-1} = a^{-1} J(S)$, and so $(R:I)_l = (R:I)_l\, R_P = a^{-1} J(S)\, R_P = a^{-1}\, R_P$. Hence we have $J(S) = R_P$, a contradiction.

Case 2. I is not finitely generated as a right S-ideal. Then, by Lemma 2.2.1, $I^{-1} = (R:I)_l$ and so $I^{-1} R_P = I^{-1}$. It follows that $R_P \subseteq O_r(I^{-1}) = O_l(I) = S$ by [MMU, (6.10)], a contradiction.

Hence there is some $x \in \mathcal{F}$ such that $O_r(SxS) = S$. The above discussion shows that there exists $x \in \mathcal{F}$ such that $A = SxSI$ is right and left $J(S)$-primary, where $O_r(SxS) = S = O_l(SxS)$. By Lemma 2.2.2, there is some $c \in SxS$ such that $SxS = cS = Sc$. Thus we have $c \in$ st(S) and $I = c^{-1}A$ ($c^{-1} \in$ st(S)).

A Goldie prime ideal P is called a *limit* prime ideal if $P = \bigcup \{ P_\lambda \mid P \supset P_\lambda :$ Goldie prime$\}$. Suppose that Q is finite dimensional over its center. Then any prime ideal is Goldie prime and it is either Archimedean or limit prime (see [BMO]). Also note that an ideal is right primary if and only if it is left primary, which is called a primary ideal (see [MMU, (13.4)]). Now we have the following proposition which describes all R-ideals in terms of primary ideals and stabilizing elements in the case Q is finite dimensional over its center.

Proposition 2.2.4. *Let R be a Dubrovin valuation ring of a simple Artinian ring Q with finite dimension over its center and I be an R-ideal with $O_r(I) = S$ ($= O_l(I)$). Then*

(1) *Suppose that I is finitely generated as a right R-ideal. Then $I = cR = Rc$ for some* $c \in$ st(R).

(2) *Suppose that I is not finitely generated as a right R-ideal.*

(a) *If $J(S)$ is Archimedean, then $I = cA = Ac$ for some $c \in$ st(S) and some $J(S)$-primary ideal A.*

(b) *If $J(S)$ is limit prime, then I is one of the following three; $I = cS = Sc$ for some $c \in$ st(S), $I = cJ(R) = J(R)c$ for some $c \in$ st(R) and $I = \bigcap c_\lambda R_\lambda$ for some $c_\lambda \in$ st(R_λ), where $R_\lambda = R_{P_\lambda}$ and P_λ runs over all Archimedean prime ideals with $P_\lambda \subset J(S)$.*

Proof. (1) Because R is Bezout, we have $I = cR$ for some $c \in$ st(R) and so $I = Rc$ by Remark in Sec. 2.1.

(2)(a) This follows from Theorem 2.2.3 and [MMU, (7.11)].

(b) First we shall prove that $J(S) = \bigcup \{P_\lambda$: Archimedean $| \ P_\lambda \subset J(S)\}$. To prove this, let x be any non-zero element in $J(S)$ and $A = SxS$. Suppose that $O_r(A) = T$. Then $A = yT = Ty$ by [MMU, (7.10)]. Thus $P = \sqrt{A}\ P_0 = \bigcap A^n$ is an Archimedean segment (see [BMO, (5)]) and $x \in P$.

Case 1. $I \subset I_v$. Then $J(R) = J(R)^2$ and $I = cJ(R)$ for some $c \in$ st(R) by Proposition 2.1.3.

Case 2. $I = I_v$. Suppose that $I \neq cS$ for any $c \in$ st(S). Then, by Lemma 1.2.8, $I = IJ(S) = I(\bigcup_\lambda P_\lambda)$. If $I = IP_\lambda$ for some λ, then $S = O_r(I) \supseteq O_r(P_\lambda) = R_{P_\lambda}$, a contradiction. So we have $IR_\lambda \supset I \supset IP_\lambda$. To show that $O_r(IR_\lambda) = R_\lambda$, suppose that $O_r(IR_\lambda) = T \supset R_\lambda$. Then $IR_\lambda = IT$ and so $IT = ITP_\lambda = IR_\lambda P_\lambda = IP_\lambda \subset I$, a contradiction. Hence $O_r(IR_\lambda) = R_\lambda$. So, by the similar method as in Lemma 1.2.8, we have $IR_\lambda = c_\lambda R_\lambda$, for some $c_\lambda \in IR_\lambda$, because $IR_\lambda = R_\lambda I$ by [MMU, (6.5) and (7.11)]. Hence $IR_\lambda = c_\lambda R_\lambda = R_\lambda c_\lambda$ by [MMU, (7.5)]. Thus $c_\lambda \in$ st(R_λ). To show that $I = \bigcap IR_\lambda$, let $B = \bigcap IR_\lambda$. Then $(R:I)_l B \subset (R:I)_l IR_\lambda = J(S)R_\lambda = R_\lambda$ for any λ by Lemma 2.1.2 (2). So $(R:I)_l B \subseteq \bigcap R_\lambda = S$ by [BMO, (4)]. Thus $B \subseteq (S:(R:I)_l)_r = (S:(S:I)_l)_r = I_v = I$ by Lemma 2.2.1 and Proposition 1.2.10, and hence $I = \bigcap c_\lambda R_\lambda$ for some $c_\lambda \in$ st(R_λ).

Remark. Proposition 2.2.4 (2)(a) is not necessarily held if Q is infinite dimensional over its center. To give a counter example, let $S = W + xT$ and $R = V + xT$ be the same as in the example of Remark in Sec. 2.1 and set $I = Sx$. Then $O_l(I) = S$ and I is not finitely generated as a left R-ideal. Assume that $I = Ac$ for some $c \in$ st(S) and some P-primary ideal A, where $P = J(S)$. Then, by [XKM, (1.10)(3)], we may assume that for some $c \in$ st(W) = K\{0}. By Remark to [XKM, (1.7)], $A = \tilde{A} + xT$ for some non-zero primary ideal $\tilde{A} = \varphi(A)$, where φ: $T = K[x,\sigma]_{(x)} \to K$ is the natural map with $\varphi(f(x)c(x)^{-1}) =$

$f_0 c_0^{-1}$ ($f(x) = f_0 + f_1 x + \ldots + f_n x^n$ and $c(x) = c_0 + c_1 x + \ldots + c_m x^m$ with $c_0 \neq 0$). So it follows that $0 = xS \cap K = cA \cap K = c\tilde{A}$, a contradiction.

CHAPTER 3

A characterization of fully bounded Dubrovin valuation rings

A ring is called *right (left) bounded* if any essential right (left) ideal contains a non-zero (two-sided) ideal. A ring is just called *bounded* if it is both right bounded and left bounded. Let S be a ring. We say that S is *fully bounded* if S/P is bounded for any prime ideal P of S. We write $J(S)$ for the Jacobson radical of S and Spec(S) for the set of all prime ideals of S.

Let R be a Dubrovin valuation ring in a simple Artinian ring Q (see [MMU, Chap.II] for the definition and elementary properties of Dubrovin valuation rings). A prime ideal P of R is called *Goldie prime* if R/P is a prime Goldie ring. We denote by G-Spec(R) the set of all Goldie prime ideals of R. Now let P_1, $P \in$ G-Spec(R) with $P_1 \supset P$. The pair $P_1 \supset P$ is called a *prime segment* if there are no Goldie primes properly between P_1 and P.

Let $P \in$ G-Spec(R) with $P \neq J(R)$ and set $P_1 = \cap\{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $P_\lambda \supset P\}$. In [BMO, (6)], they have shown that the following four cases only occur:

(1) P is lower limit, i.e., $P = P_1$. Otherwise, $P_1 \supset P$ is a prime segment.

(2) $P_1 \supset P$ is Archimedean.

(3) $P_1 \supset P$ is simple.

(4) $P_1 \supset P$ is exceptional, i.e., there exists a non-Goldie prime ideal C such that $P_1 \supset C \supset P$.

In section 3.1, we prove that R is fully bounded iff (1) and (2) only hold (see Theorem 3.1.5). (Note that $R/J(R)$ is bounded, because it is a simple Artinian ring). For any regular element c in $J(R)$, we define $P(c) = \cap\{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $c \in P_\lambda\}$, a Goldie prime ideal ([BMO, (1)]). R is called *locally invariant* if $cP(c) = P(c)c$ for any regular element c in $J(R)$. This concept was defined by Gräter [G] in order to study the approximation theorem in the case where R is a total valuation ring. We show that R is fully bounded if and only if it is locally invariant, by using Theorem 3.1.5 (see Proposition 3.1.6).

In section 3.2, we give several examples of of fully bounded Dubrovin valuation rings of Q with infinite dimension over its center. If Q is of finite dimensional over its center, then R is always fully bounded.

3.1. Fully bounded Dubrovin valuation rings

Throughout this section, R will denote a Dubrovin valuation ring in a simple Artinian ring Q. For any $P \in$ Spec(R), set $C(P) = \{c \in R \mid c$ is regular mod $P\}$. If $P \in$ G-Spec(R), then $C(P)$ is localizable and we denote by R_P the localization of R at P. Before starting the lemmas, we note the following: there is a one-to-one correspondence between G-Spec(R) and the set of all overrings of R, which is given by $P \to R_P$ with $P = J(R_P)$ and $S \to J(S)$ ($P \in$ G-Spec(R) and S is an overring of R). Furthermore, for any P_1, $P \in$ G-Spec(R), $P_1 \supset P$ iff $R_P \subset R_{P_1}$ ([MMU, (§ 6)] and [BMO, (§ 2)]). We will use these properties throughout the chapter.

Lemma 3.1.1. *Let S be an order in Q and A be an S-ideal such that $O_r(A) = T = O_l(A)$ where $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$ and $O_l(A) = \{g \in Q \mid gA \subseteq A\}$. Suppose that $A = aT$ for some $a \in A$. Then $A = Ta$.*

Proof. $T = O_l(A) = aTa^{-1}$ implies $A = Ta$.

Lemma 3.1.2. *Let R be a Dubrovin valuation ring of Q and $P \in$ G-Spec(R). Suppose that P is lower limit, i.e., $P = \cap\{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $P_\lambda \supset P\}$. Then $R_P = \cup R_{P_\lambda}$ and $C(P) = \cup C_{P_\lambda}$.*

Proof. Since $P_\lambda \supset P$, it follows that $R_P \supset R_{P_\lambda}$ so that $R_P \supseteq S = \cup R_{P_\lambda}$. Suppose that $R_P \supset S$. Then for any P_λ, $P_\lambda = J(R_{P_\lambda}) \supseteq J(S) \supset J(R_P) = P$ implies $P = \cap P_\lambda \supseteq J(S) \supset P$, a contradiction. Hence $R_P = \cup R_{P_\lambda}$ and so $C(P) = \cup C(P_\lambda)$ follows.

Lemma 3.1.3. *Let R be a Dubrovin valuation ring of Q and* $P \in$ G-Spec(R). *Then*

(1) Spec(R_P) = { P_1 | $P_1 \in$ Spec(R) *with* $P \supseteq P_1$ }.

(2) *Let* P_1 *and* P_2 *be in* Spec(R) *with* $P \supseteq P_1 \supset P_2$. *Then* $P_1 \supset P_2$ *is a prime segment of R if and only if it is a prime segment of* R_P.

Proof. (1) Let $P_1 \in$ Spec(R_P).

Case 1. If P_1 is Goldie prime, then $(R_P)_{P_1}$ is an overring of R_P (and so of R) with $J((R_P)_{P_1}) = P_1$, i.e., $P_1 \in$ Spec(R) and $P = J(R_P) \supseteq P_1$.

Case 2. If P_1 is non-Goldie prime, then we can construct an exceptional prime segment of R_P, say $P_2 \supset P_1 \supset P_0$ by [BMO, (6)]. By case 1, $P \supseteq P_2$ and P_2, $P_0 \in$ G-Spec(R). It easily follows from note before Lemma 3.1.1 that there are no Goldie primes properly between P_2 and P_0, which implies $P_2 \supset P_0$ is a prime segment of R. As in [BMO], let $K(P_2) = \{a \in P_2 \mid P_2 a P_2 \subset P_2\}$. Then $K(P_2) = P_1$ by [BMO, (7)] and so $P_2 \supset P_0$ is an exceptional prime segment of R with $K(P_2) = P_1$, i.e., P_1 is non-Goldie prime of R with $P \supset P_1$. Conversely, let $P_1 \in$ Spec(R) with $P \supseteq P_1$. Then from note before Lemma 3.1.1 and the method we have just done, we can easily see that $P_1 \in$ Spec(R_P) and that $P_1 \in$ G-Spec(R) iff $P_1 \in$ G-Spec(R_P).

(2) This is clear from (1).

Lemma 3.1.4. *Let R be a Dubrovin valuation ring of Q and* $P_1 \supset P$ *be an Archimedean prime segment. Then for any* $c \in P_1 \backslash P$, *the following hold:*

(1) $R_{P_1} c R_{P_1} = a R_{P_1} = R_{P_1} a$, *for some* $a \in P_1$.

(2) *If c is a regular element, then* $c R_{P_1} = R_{P_1} c$ and $c P_1 = P_1 c$.

Proof. Firstly note that $P_1 \supset P$ is an Archimedean prime segment of R_{P_1} by Lemma 3.1.3 and [BMO, (7)].

(1) Let $\tilde{R}_{P_1} = R_{P_1}/P$, a Dubrovin valuation ring of $\overline{R_P} = R_P/P$ (see Theorem 1.2.6) such that $J(\tilde{R}_{P_1}) = \tilde{P}_1 = P_1/P$ and $\tilde{P}_1 \supset (\tilde{0})$ is Archimedean. Here for any $a \in R_{P_1}$, we

write $\tilde{a}$ for the image of a in $\tilde{R}_{P_1}$. If $\tilde{P}_1 = \tilde{P}_1^{\,2}$, then $\tilde{0} \neq \tilde{R}_{P_1} \tilde{c} \tilde{R}_{P_1} = \tilde{a} \tilde{R}_{P_1} = \tilde{R}_{P_1} \tilde{a}$ for some $a \in P_1$ by [BMU, (2.1)]. If $\tilde{P}_1 \supset \tilde{P}_1^{\,2}$, then $\tilde{R}_{P_1}$ is a Noetherian Dubrovin valuation ring and so any ideal of $\tilde{R}_{P_1}$ is power of $\tilde{P}_1$. Thus $\tilde{R}_{P_1} \tilde{c} \tilde{R}_{P_1} = \tilde{a} \tilde{R}_{P_1} = \tilde{R}_{P_1} \tilde{a}$ for some $a \in P_1$, because $\tilde{P}_1$ is principal. Hence, in both cases, $R_{P_1} c R_{P_1} + P = aR_{P_1} + P = R_{P_1} a + P$. However, since $\tilde{a} \in C_{\tilde{R}_{P_1}}(\tilde{0}) = \{\tilde{b} \in \tilde{R}_{P_1} \mid \tilde{b}$ is regular in $\tilde{R}_{P_1}\}$, it follows that $a \in C_{R_{P_1}}(P)$ and so a is a regular element by Proposition 1.3.2. Thus we have $aR_{P_1} a^{-1} \subseteq aR_P a^{-1} = R_P$. It follows that aR_{P_1} and P are both left $aR_{P_1} a^{-1}$ and right R_{P_1}-ideals. Hence $aR_{P_1} \supset P$ by [MMU, (6.4)] and similarly $R_{P_1} a \supset P$. Since $R_{P_1} c R_{P_1}$ and P are both ideals of R_{P_1}, it follows that $R_{P_1} c R_{P_1} \supset P$. Therefore $R_{P_1} c R_{P_1} = aR_{P_1} = R_{P_1} a$ follows.

(2) By (1), $P_1 \supseteq R_{P_1} c R_{P_1} = R_{P_1} a = aR_{P_1}$ for some $a \in P_1$. Suppose that $cR_{P_1} \subset R_{P_1} c R_{P_1}$. Then, by Lemma 1.2.4, there is a $b \in R_{P_1} c R_{P_1}$ such that $cR_{P_1} \subseteq bP_1 \subseteq aP_1$, because $Q_l(cR_{P_1}) = cR_{P_1} c^{-1}$ and $P_1 = J(R_{P_1})$. So $R_{P_1} a^{-1} c R_{P_1} \subseteq P_1$. On the other hand, $R_{P_1} c R_{P_1} = aR_{P_1}$ implies that $R_{P_1} a^{-1} c R_{P_1} = R_{P_1}$, a contradiction. Hence, $cR_{P_1} = R_{P_1} c R_{P_1}$ and similarly $R_{P_1} c = R_{P_1} c R_{P_1}$ so that $cR_{P_1} = R_{P_1} c$. Since $cR_{P_1} c^{-1} = R_{P_1}$ and $J(R_{P_1}) = P_1$, we have $cP_1 c^{-1} = P_1$ and so $cP_1 = P_1 c$.

Theorem 3.1.5. *Let R be a Dubrovin valuation ring of a simple Artinian ring Q. Then R is fully bounded if and only if for any $P \in$ Spec(R), $P \neq J(R)$, the following hold:*

(1) $P \in$ G-Spec(R).

(2) *P is either lower limit or there is a $P_1 \in$ Spec(R) such that $P_1 \supset P$ is an Archimedean prime segment.*

Proof. Suppose that R is fully bounded.

(1) Assume that there is a non-Goldie prime ideal C. Then we have an exceptional prime segment, say, $P_1 \supset C \supset P_2$ by [BMO, (6)]. R is an n-chain ring by Theorem

1.2.2 and so is $\bar{R} = R/C$. This implies that $\bar{R}$ has a finite Goldie dimension, say, m ($\leq n$). Thus there are non-zero uniform right ideals $\bar{U_i}$ of $\bar{R}$ such that $\bar{U_1} \oplus \ldots \oplus \bar{U_m}$ is an essential right ideal of $\bar{R}$. Since $\bar{R}$ is a prime ring, $\bar{U_i} \cap \bar{P_1} \supseteq \bar{U_i}\,\bar{P_1} \neq \bar{0}$ and so there are non-zero $\bar{u_i} \in \bar{U_i} \cap \bar{P_1}$, where $u_i \in P_1$. Set $I = u_1R + \ldots + u_mR$. Then $I = aR$ for some $a \in I$, because R is Bezout (Theorem 1.2.2) and $\bar{I} = \bar{u_1}\,\bar{R} \oplus \ldots \oplus \bar{u_m}\,\bar{R} = \bar{a}\,\bar{R}$ is an essential right ideal of $\bar{R}$. We claim that $\bar{P_1} \supset \bar{I}$. On the contrary, suppose that $\bar{P_1} = \bar{I}$, i.e., $P_1 = aR + C$. Note that $O_l(C) = R_{P_1} = O_r(C)$ by [BMU, (2.2)] so that C is an ideal of R_{P_1}. If C is a principal right ideal of R_{P_1}, say, $C = cR_{P_1}$ for some $c \in C$, then $P_1 = aR_{P_1} + cR_{P_1} = bR_{P_1}$ for some $b \in P_1$. It follows from Lemma 3.1.1 that $P_1 = bR_{P_1} = R_{P_1}b$ and so $P_1 \supset P_1^{\,2} \supset C$, which contradicts to the fact that there are no ideals properly between P_1 and C (cf. [BMO, (6)]). If C is not a principal right ideal of R_{P_1}, then $CP_1 = C$ by Lemma 1.2.8 and so $P_1 = P_1^{\,2} = aP_1 + CP_1 = aP_1 + C$. Thus we have $a = ap + d$ for some $p \in P_1$ and $d \in C$ and $a(1-p) = d \in C$. It follows that $a \in C$, because $1-p$ is a unit of R_{P_1}, which shows $\bar{I} = \bar{0}$, a contradiction. We have shown that $\bar{P_1} \supset \bar{I}$ and $\bar{I}$ is an essential right ideal of $\bar{R}$. Hence $\bar{R}$ is not bounded, because there are no ideals properly between P_1 and C. Therefore, any prime ideal of R is Goldie prime.

(2) Let $P \in$ G-Spec(R) and suppose that P is not lower limit. Then there is a $P_1 \in$ G-Spec(R) such that $P_1 \supset P$ is a prime segment, which is not exceptional by (1). Suppose that this is simple. For any $c \in P_1 \cap C(P)$, it follows that $\bar{c}\,\bar{P_1}$ is an essential right ideal of $\bar{R} = R/P$, which is a Dubrovin valuation ring of R_P/P (Corollary 1.2.3). Suppose that $\bar{c}\,\bar{P_1} = \bar{P_1}$, i.e., $cP_1 + P = P_1$. Since cP_1 and P are both left $cR_{P_1}c^{-1}$ and right R_{P_1}-ideals (note $cR_{P_1}c^{-1} \subseteq cR_Pc^{-1} = R_P$), we have either $cP_1 \supset P$ or $cP_1 \subseteq P$ by Proposition 1.2.5. The latter case is impossible and so $cP_1 \supset P$. Thus $cP_1 = P$ and $c^{-1} \in O_l(P_1) = R_{P_1}$ follows. This is contradiction, because $c \in P_1$. Hence we have shown that $\bar{P_1} \supset \bar{c}\,\bar{P_1}$ and $\bar{c}\,\bar{P_1}$ is an essential right ideal. Therefore, $\bar{R}$ is not bounded,

because there are no ideals properly P_1 and (0). Hence either P is lower limit or there is a $P_1 \in$ G-Spec(R) such that $P_1 \supset P$ is an Archimedean prime segment.

Conversely, suppose that the conditions (1) and (2) hold and let $P \in$ Spec(R). Then P is Goldie prime by (1). Firstly, assume that P is lower limit, i.e., $P = \cap\{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $P_\lambda \supset P\}$. Then $C(P) = \cup\, C(P_\lambda)$ by Lemma 3.1.2. So, for any $c \in C(P)$, we have $c \in C(P_\lambda)$ for some λ. Then $cR \supset P_\lambda$, because cR and P_λ are both left cRc^{-1} and right R-ideals. Hence $\bar{c}\,\bar{R} \supset \overline{P_\lambda} \neq \bar{0}$ in $\bar{R} = R/P$, showing that $\bar{R}$ is bounded. Secondly, suppose that the prime segment $P_1 \supset P$ is Archimedean and let $c \in C(P)$. Then, as before, $\bar{c}\,\overline{P_1}$ is an essential right ideal of $\bar{R} = R/P$ and so $cP_1 \cap C(P) \neq \varnothing$. Let $d \in cP_1 \cap C(P)$. Then, by Lemma 3.1.4 (2) and Theorem 1.3.3, $cR \supseteq cP_1 \supseteq dR_{P_1} = R_{P_1}d$ and $dR_{P_1} \supset P$ follows. Therefore, $\bar{R} = R/P$ is bounded and hence R is fully bounded.

As an application of Theorem 3.1.5, we have the following:

Proposition 3.1.6. *Let R be a Dubrovin valuation ring of a simple Artinian ring Q. Then R is locally invariant if and only if it is fully bounded.*

Proof. Suppose that R is locally invariant. In order to prove that it is fully bounded, on the contrary, assume that R is not fully bounded. Then there are prime ideals P, P_1 such that either the prime segment $P_1 \supset P$ is simple or $P_1 \in$ G-Spec(R), P is a non-Goldie prime ideal and there are no ideals properly between P_1 and P. In either case, we shall prove that there is a regular element $c \in P_1 \setminus P$. Let c_1 be any element in $P_1 \setminus P$. If c_1R is an essential right ideal, then $c = c_1$ is regular. If c_1R is not an essential right ideal, then there is a right ideal I such that $cR \oplus I$ is essential. So it follows from Goldie's theorem that $(cR \oplus I)P_1$ is also an essential right ideal which is contained in P_1 but not in P. So there is a regular element $c \in (c_1R \oplus I)\ P_1$ but not in P by [MR, (3.3.7)]. Now let $c \in P_1 \setminus P$ such that c is regular. Then $cP_1 = P_1c$, because $P_1 = P(c)$.

Since $P_1 \supseteq cP_1 = P_1 c \supset P$, we have $cP_1 = P_1$, which implies $c^{-1} \in O_l(P_1) = R_{P_1}$. Hence $R_{P_1} = cR_{P_1} \subseteq P_1$, a contradiction. Therefore, R is fully bounded.

Suppose that R is fully bounded. Let $c \in J(R)$ such that c is regular. By the assumption and Theorem 3.1.5, $P(c) = \cap\{P_\lambda \mid P_\lambda \in \text{Spec}(R)$ such that $P_\lambda \ni c\}$, which is Goldie prime by Proposition 1.2.13. Suppose that $P(c)$ is upper limit, i.e., $P(c) = \cup\{P_\mu \mid P_\mu \in \text{G-Spec}(R)$ such that $P_\mu \subset P(c)\}$. Then there is a P_μ with $P_\mu \ni c$. This contradicts the choice of $P(c)$. Hence $P(c) \supset P = \cup\{P_\mu \mid P(c) \supset P_\mu\}$ is a prime segment which must be Archimedean by Theorem 3.1.5. Since $c \in P(c) \setminus P$ and c is regular, we have $cP(c) = P(c)c$ by Lemma 3.1.4. Hence R is locally invariant.

We say that R is *invariant* if $cRc^{-1} = R$ for any regular element c in R and that it is of *rank n* if there are exactly n Goldie prime ideals. From Lemma 3.1.4, we have

Proposition 3.1.7. *Suppose that R is Archimedean and is of rank one. Then it is invariant.*

Proof. Let c be any regular element and let c_1 be any regular element in $J(R)$. Then we have $cRc^{-1} = cc_1R(cc_1)^{-1} = R$ by Lemma 3.1.4, because $c_1, cc_1 \in J(R)$.

3.2. Examples

We will give several examples of fully bounded Dubrovin valuation rings.

Example 3.2.1. Any Dubrovin valuation ring of a simple Artinian ring with finite dimension over its center is fully bounded.

Example 3.2.2. Any invariant valuation ring of a division ring is fully bounded (see [XKM, (Remarks to Examples 2.1 and 2.4)] for invariant valuation rings of division rings with infinite dimensions over its centers).

In order to give more general examples, we recall the skew polynomial ring $Q[x,\sigma]$ over Q in an indeterminate x, where $\sigma \in \mathrm{Aut}(Q)$. Since $Q[x,\sigma]$ is a principal ideal ring, the maximal ideal $P = xQ[x,\sigma]$ is localizable, i.e., $T = Q[x,\sigma]_P = \{f(x)c(x)^{-1} \mid f(x) \in Q[x,\sigma]$ and $c(x) \in C(P)\}$, the localization of $Q[x,\sigma]$ at P, is a Noetherian Dubrovin valuation ring with $J(T) = xT$. Since Q is a simple Artinian ring, $C(P) = \{c(x) \in Q[x,\sigma] \mid c(x) = c_0 + c_1x + \ldots + c_nx^n$ such that c_0 is a unit in $Q\}$. For any $t = f(x)c(x)^{-1} \in T$, where $f(x) = f_0 + f_1x + \ldots + f_lx^l$ and $c(x) = c_0 + c_1x + \ldots + c_nx^n$, the map φ: $T \to Q$ defined by $\varphi(t) = f_0c_0^{-1}$ is a ring epimorphism. Now let R be a Dubrovin valuation ring of Q. Then, by [XKM, (1.6)], $\tilde{R} = \varphi^{-1}(R)$, the complete inverse image of R by φ, is a Dubrovin valuation ring of $Q(x,\sigma)$ ($Q(x,\sigma)$ stands for the quotient ring of $Q[x,\sigma]$). Furthermore, let $P = \wp\tilde{R}$ ($\wp \in \mathrm{Spec}(R)$). Then $P \in \mathrm{Spec}(\tilde{R})$ and $\tilde{R}/P \cong R/\wp$ by [XKM, (1.6)] and its proof. Thus it follows from [XKM, (1.6)] that $\tilde{R}$ is fully bounded iff R is fully bounded. Hence we have

Example 3.2.3. With notation above, suppose that R is a fully bounded Dubrovin valuation ring of Q and that σ is of infinite order ([XKM, (Examples 2.1 – 2.6, 2.7 and 2.8)]). Then $\tilde{R}$ is a fully bounded Dubrovin valuation ring of $Q(x,\sigma)$ and $Q(x,\sigma)$ is of infinite dimensional over the center.

Finally, we give a few remarks on non-fully bounded total valuation rings: An example of a total valuation ring with a simple segment was first constructed by [Mt]. See [BT] for other examples of total valuation rings with simple segments. Dubrovin constructed an example of a total valuation ring with an exceptional prime segment ([D3]).

CHAPTER 4

Non-commutative ν-Bezout rings

Throughout this chapter, V will be a total valuation ring of a division ring K, i.e., for any nonzero $k \in K$, either $k \in V$ or $k^{-1} \in V$. Let Q_0 be the semigroup of nonnegative rational numbers and σ be a semigroup homomorphism from Q_0 to Aut(V), the group of automorphism of V, i.e., $\sigma(r + s) = \sigma(r).\sigma(s)$ for any $r,s \in Q_0$. Furthermore, $R = V[x^r, \sigma \mid r \in Q_0]$ is a skew semigroup ring of Q_0 over V, i.e., it is a ring with left V-basis $\{x^r \mid r \in Q_0\}$. Each element of R is uniquely a finite sum $a_1x^{r_1} + \ldots + a_kx^{r_k}$ with $a_i \in V$. The multiplication is defined by $x^r a = \sigma(r)(a)x^r$ for any $a \in V$ and $r \in Q_0$. Since σ is naturally extended to a semigroup homomorphism from Q_0 to Aut(K), we have $T = K[x^r, \sigma \mid r \in Q_0]$ is a skew semigroup ring of Q_0 over K.

In Section 1, we prove that $R = V[x^r, \sigma \mid r \in Q_0]$ is ν-Bezout, which is defined in [Ma] and is a non-commutative version of commutative GCD-domains.

In Section 2, we give some examples of non-commutative ν-Bezout rings with some types of automorphisms.

4.1. Non- commutative ν-Bezout rings

Let S be an Ore domain with its quotient ring Q and let $I(J)$ be a right (left) S-ideal. We use the following notation [MMU]: $(S\colon I)_l = \{q \in Q \mid qI \subseteq S\}$, $(S\colon J)_r = \{q \in Q \mid Jq \subseteq S\}$, $I_v = (S\colon (S\colon I)_l)_r$ and ${}_vJ = (S\colon (S\colon J)_r)_l$. It is clear that I_v (${}_vJ$) is a right (left) S-ideal containing $I(J)$, respectively. If $I = I_v$ ($J = {}_vJ$), then it is called a *right (left) v-ideal.* An Ore domain S is called *right v-Bezout* if I_v is a principal for any finitely generated right ideal I of S. Similarly, we can define *left v-Bezout* and S is said to be *v-Bezout* if it is right ν-Bezout as well as left ν-Bezout.

A partially ordered set Λ with ordering $\geq$ is called an *ascending net* if for any λ_1, λ_2 in Λ, there is a $\lambda \in \Lambda$ with $\lambda_i \leq \lambda$ (i = 1, 2). Then we have the following lemma.

Lemma 4.1.1. *Let Λ be an ascending net and let R_λ be an Ore domain with its quotient division ring K_λ, for each $\lambda \in \Lambda$. Suppose that $R_\mu \subseteq R_\lambda$ if $\mu \leq \lambda$. Set $R = \cup\{R_\lambda \mid \lambda \in \Lambda\}$ and $K = \cup\{K_\lambda \mid \lambda \in \Lambda\}$. Then*

(1) *K is a quotient ring of R which is a division ring.*

(2) *If R_λ is a Bezout ring for all $\lambda \in \Lambda$, then so is R.*

(3) *Let P_λ be a completely prime ideal of R_λ, which is localizable for any $\lambda \in \Lambda$. Suppose that $P_\lambda \cap R_\mu = P_\mu$ if $\lambda \geq \mu$. Then*

   (a) *$P = \cup\{P_\lambda \mid \lambda \in \Lambda\}$ is a completely prime ideal of R and is localizable.*

   (b) *$R_P = \cup\{R_{\lambda_{P_\lambda}} \mid \lambda \in \Lambda\}$.*

   (c) *If $R_{\lambda_{P_\lambda}}$ is a total valuation ring for all $\lambda \in \Lambda$, then so is R_P.*

Let N be the set of natural numbers. Then it is considered an ascending net in the following obvious way: $n \geq m$ iff $m \mid n$ for any $m, n \in$ N. Let $R_n = V[x^{\frac{1}{n}}, \sigma] = \{a_k x^{\frac{k}{n}} + \ldots + a_1 x^{\frac{1}{n}} + a_0 \mid a_i \in V\}$. Then R_n is considered as a skew polynomial ring over V in the indeterminate $x^{\frac{1}{n}}$ with $x^{\frac{1}{n}} a = \sigma(\frac{1}{n})(a) x^{\frac{1}{n}}$ for any $a \in V$. Let $P_n = J(V)[x^{\frac{1}{n}}, \sigma]$, a completely prime ideal of R_n and it is localizable such that $R_{n_{P_n}}$ is a total valuation ring of $K(x^{\frac{1}{n}}, \sigma)$ (see [BT]). Obviously, $R_n \supseteq R_m$ and $P_m = P_n \cap R_m$ if $n \geq m$. Furthermore, $P = J(V)[x^r, \sigma \mid r \in Q_0] = \cup\{P_n \mid n \in \text{N}\}$. Let $R = V[x^r, \sigma \mid r \in Q_0]$, $T = K[x^r, \sigma \mid r \in Q_0]$ and let $T_n = K[x^{\frac{1}{n}}, \sigma]$, be a principal ideal ring for each $n \in$ N. Then it is obvious that $R = \bigcup_{n\in \mathbb{N}} R_n$ and $T = \bigcup_{n\in \mathbb{N}} T_n$. So from Lemma 4.1.1, we have the following:

Proposition 4.1.2. (1) $P = J(V)[x^r, \sigma \mid r \in \mathbb{Q}_0]$ *is localizable and* R_P *is a total valuation ring with* $R_P = \cup R_{n_{P_n}}$.

(2) $T = K[x^r, \sigma \mid r \in \mathbb{Q}_0]$ *is a Bezout ring with its quotient ring* $K(x^r, \sigma \mid r \in \mathbb{Q}_0)$.

Let δ be a left τ-derivation of V, where $\tau \in \mathrm{Aut}(V)$ and assume that (τ, δ) is *compatible*, i.e., $\delta(J(V)) \subseteq J(V)$. Let $S = V[x; \tau, \delta]$ be an Ore extension over V in an indeterminate x. Then $P = J(V)[x; \tau, \delta]$ is localizable and S_P, the localization of S at P, is a total valuation ring (cf. [BT]). Now let $f(x), g(x) \in S$ and let $I = Sf(x) + Sg(x)$. Then $S_P I = S_P a$, for some $a \in V$ and $K[x; \tau, \delta] I = K[x; \tau, \delta] b(x)$, for some $b(x) \in K[x; \tau, \delta]$. There are $b \in K$ and $b_1(x) \in S \setminus P$ with $b(x)a^{-1} = b b_1(x)$. With these notations, we have the following:

Lemma 4.1.3. [Ma, (2.1) and (2.3)]. ${}_vI = S_P I \cap K[x; \tau, \delta] I = Sc(x)$, where $c(x) = b_1(x) a \in S$.

By using Lemma 4.1.3, we have the following theorem which is inspired by [C, (3.5)].

Theorem 4.1.4. *Let V be a total valuation ring of a division ring K. Then $R = V[x^r, \sigma \mid r \in \mathbb{Q}_0]$ is v-Bezout, and it is not Bezout if $V \neq K$.*

Proof. Let $I = Rf(x) + Rg(x)$, for some $f(x), g(x) \in R$. There is a natural number m such that $f(x), g(x) \in R_m$. Set $I_m = R_m f(x) + R_m g(x)$. Then by Lemma 4.1.3, there is $c(x) \in R_m$ with $R_{m_{P_m}} I_m \cap T_m I_m = R_m c(x)$, where $R_{m_{P_m}} I_m = R_{m_{P_m}} a$ ($a \in V$), $T_m I_m = T_m b(x)$ ($b(x) \in T_m$), $b(x)a^{-1} = b b_1(x)$ ($b \in K$, $b_1(x) \in R_m \setminus P_m$) and $c(x) = b_1(x) a$. For any natural number n with $m \mid n$, we have $R_{n_{P_n}} I_n = R_{n_{P_n}} a$ and $T_n I_n = T_n b(x)$ and so ${}_vI_n = R_{n_{P_n}} I_n \cap T_n I_n = R_n c(x)$ by Lemma 4.1.3. Since $f(x), g(x) \in R_m c(x) \subseteq Rc(x)$, it follows that $I \subseteq Rc(x)$. Suppose that $I \subseteq R\alpha$ for some $\alpha \in K(x^r, \sigma \mid r \in Q_0)$. Then $\alpha \in K(x^{\frac{1}{n}}, \sigma)$, the quotient ring of T_n, for some n and we assume that $m \mid n$. It follows that $I_n \subseteq R\alpha \cap K(x^{\frac{1}{n}}, \sigma) = R_m \alpha$ and so $R_n c(x) = {}_v I_n \subseteq R_n \alpha$.. Thus $Rc(x) \subseteq R\alpha$ and hence ${}_v I = Rc(x)$ follows. If $J = J_1 + J_2$, where J_1 and J_2 are left ideals of R, then it is easy to check that ${}_v J = {}_v({}_v J_1 + J_2) = {}_v({}_v J_1 + {}_v J_2)$ and so R is left v-Bezout by induction on generators. Similarly, R is right v-Bezout.

Now, suppose that $R = V[x^r, \sigma \mid r \in Q_0]$ is left Bezout. Let α be a non-unit element in $V \setminus \{0\}$. Then there exists $h(x) \in R$ such that $R\alpha + Rx = Rh(x)$. We have $\alpha = a(x)\, h(x)$ and $x = b(x)\, h(x)$ for some $a(x), b(x) \in R$. Then it follows that $h(x)$ is constant, say, $h(x) = c$ and $b(x) = b_1 x$ for some $b_1 \in V$. Thus $1 = b_1 \sigma(1)(c)$ and so c is unit in V. Then $R\alpha + Rx = Rh(x) = Rc = R$ implies that α is unit in V, a contradiction. Hence R is not left Bezout.

4.2. Examples

Finally, we will give several examples of skew semigroup ring of Q_0 over total valuation rings.

Example 4.2.1 (trivial case, $\sigma = 1$). $R = V[x^r \mid r \in Q_0]$ is v-Bezout, where V is any total valuation ring.

In order to provide non-trivial examples, let $K = F(\{Y_t\} \mid t \in Q)$ be the rational function field over a field F in indeterminates $\{Y_t \mid t \in Q\}$, where Q is the field of rationals. For any $r \in Q_0$, let $\sigma_r \in \mathrm{Aut}(K)$ determined by; $\sigma_r(a) = a$ for any $a \in F$ and $\sigma_r(Y_t) = Y_{t+r}$ for any $t \in Q$. Furthermore, let v be the valuation of K determined by $v(a) = 0$ for all $a \in F$ and $v(Y_t) = 1$ for all $t \in Q$. Then $V = \{k \in K \mid v(k) \geq 0\}$ is a discrete rank one valuation ring of K. It is easy to see that $\sigma_r(V) = V$ for any $r \in Q_0$

and $\sigma_{r+s} = \sigma_r . \sigma_s$. Hence the mapping $\sigma : Q_0 \to \mathrm{Aut}(V)$ defined by $\sigma(r) = \sigma_r$ for any $r \in Q_0$ is a semigroup homomorphism.

Example 4.2.2. With the notation and assumption the above, $R = V[x^r \mid r \in Q_0]$ is v-Bezout which is not Bezout.

In order to get another example which is not discrete rank one valuation ring, let $G = \oplus Z_r (r \in Q, Z_r = Z)$, the direct sum of the copies Z, which is a totally ordered abelian group by lexicographic ordering and let K and σ_r be as in Example 4.2.2. We define a valuation of K as follows: $v(a) = 0$ for all $a \in F$ and $v(Y_t) = (\ldots, 0, 1, 0, \ldots) \in G$, the t-th component is 1 and the other components are all zero. Then $V = \{k \in K \mid v(k) \geq 0\}$ is a valuation ring of K with infinite rank and $J(V) = J(V)^2$. It is not hard to see that $\sigma_r(V) = V$ for all $r \in Q_0$. Hence, we have

Example 4.2.3. $R = V[x^r \mid r \in Q_0]$ is v-Bezout, where V is commutative valuation ring with infinite rank and $J(V) = J(V)^2$.

In order to give an example of non-commutative valuation rings, let be V_0 be any total valuation ring of a division ring K_0 and $G = \langle g_r \mid r \in Q\rangle$ be a group which is isomorphic to Q, i.e., $g_r . g_s = g_{r+s}$ for any $r, s \in Q$. Since G is abelian, the group ring $V_0[G]$ and $K_0[G]$ have the same quotient ring $K_0(G)$ which is a division ring. As before, for any $r \in Q_0$ we define an automorphism σ_r of $K_0(G)$ as follows: $\sigma_r(a) = a$ for all $a \in K_0$ and $\sigma_r(g_t) = g_{t+r}$ for any $t \in Q$. Now $J(V_0)[G]$ is localizable and $V = V_0[G]_{J(V_0)[G]}$ is a total valuation ring of $K_0(G)$ (see [BMO, (2.6)]). Since $\sigma_r(J(V)[G]) = J(V)[G]$, σ_r is considered as an automorphism of V with $\sigma_{r+s} = \sigma_r . \sigma_s$ for any $r, s \in Q_0$. So the mapping $\sigma : Q_0 \to \mathrm{Aut}(V)$ given by $\sigma(r) = \sigma_r$ for any $r \in Q_0$ is a semigroup homomorphism.

Example 4.2.4. With the notation and assumption the above, $R = V[x^r \mid r \in Q_0]$ is v-Bezout but not Bezout.

CHAPTER 5

Overrings of Non-commutative Prüfer rings satisfying a polynomial identity

In [AD], they defined the concept of non-commutative Prüfer rings in the context of prime Goldie rings and studied the structure of Prüfer rings. In the case when prime rings satisfying a polynomial identity (PI), Morandi studied PI Prüfer rings under some conditions such as; integral over its center or the center is commutative Prüfer. Furthermore, Dubrovin [D_2] proved that any prime ideal of a PI Prüfer ring is localizable.

In Section 1, we describe the properties of overrings of PI Prüfer rings.

In Section 2, we describe prime ideals of any overring of a PI Prüfer ring by using some results in [D_2] and [Mo].

We refer the readers to [MMU] for elementary properties of Prüfer rings and Dubrovin valuation rings.

5.1. Overrings of PI Prüfer rings

Throughout this chapter, R will be a prime Goldie ring with its quotient ring Q. Let I be an additive subgroup of Q. Then the *right* and *left orders* of I are defined to be $O_r(I) = \{ q \in Q \mid Iq \subseteq I \}$, and $O_l(I) = \{ q \in Q \mid qI \subseteq I \}$. We also define $I^{-1} = \{ q \in Q \mid IqI \subseteq I \}$), the *inverse* of I. If I is a right R-submodule of Q, then I is a (fractional) *right R-ideal* if I contains a regular element of Q, and if there is a regular element $d \in Q$ with $dI \subseteq R$. Left R-ideals are defined similarly.

Following [AD], R is called *right Prüfer* if for every finitely generated right R-ideal I, $I^{-1}I = R$, $I I^{-1} = O_l(I)$. A left Prüfer ring is defined similarly. It is proved in [AD, (1.12)] that R is right Prüfer if and only if it is left Prüfer. A ring is called *right (left) Bezout* is any finitely generated right (left) ideal is principal. We say that R is a *Dubrovin valuation ring* if R is Bezout and $R/J(R)$ is a simple Artinian ring, where $J(R)$ is the Jacobson radical of R. A prime ideal P of R is said to be *localizable* if $C(P) = \{c \in R \mid c$ is regular mod $P\}$ is an Ore set of R. Let P be a non-zero prime ideal of a PI

Prüfer ring R. Then any element of $C(P)$ is regular, $C(P)$ is localizable and R_P is a Dubrovin valuation ring ([D2]). We write Spec(R) for the set of all prime ideals of R.

Lemma 5.1.1. *Let S be an overring of R. Suppose that S is flat as a left R-module. Then $S \otimes_R S \cong S$ naturally.*

Proof. For any $\alpha = \sum s_i \otimes t_i$, where s_i, $t_i \in S$, we define $\varphi(\alpha) = \sum s_i t_i$. Then there is a regular element $c \in R$ with $s_i = c^{-1}\overline{s_i}$ for some $\overline{s_i} \in R$. From the exact sequence $0 \to S \to Q$, we derive the exact sequence $0 \to S \otimes_R S \to Q \otimes_R S$. Then $\alpha = \sum s_i \otimes t_i = \sum c^{-1}\overline{s_i} \otimes t_i = c^{-1} \otimes \sum \overline{s_i}\, t_i \in Q \otimes_R S$. So if $\varphi(\alpha) = 0$, then $0 = \sum s_i t_i = c^{-1}(\sum \overline{s_i}\, t_i)$ and thus $\alpha = 0$, which shows that φ is one-to-one. It is clear that φ is onto and hence φ is an isomorphism.

Let I be a right ideal of R and $s \in Q$. Then we use the following notation; $s^{-1}I = \{r \in R \mid sr \in I\}$ which is a right ideal of R.

Lemma 5.1.2. *Under the same notation and assumption as in Lemma 5.1.1, let I be a non-zero right ideal of R and let $s \in S$, non-zero. Then $(s^{-1}I)S = s^{-1}(IS) = \{t \in S \mid st \in IS\}$.*

Proof. It is clear that $(s^{-1}I)S \subseteq s^{-1}(IS)$. To prove the converse inclusion, we consider the exact sequence $0 \to s^{-1}I \to R \xrightarrow{s_l} S/I$, where $s_l(r) = [sr + I]$ for all $r \in R$. Then since S is a flat left R-module, we have the following exact sequence:

$$0 \to (s^{-1}I) \otimes_R S \to R \otimes_R S \xrightarrow{s_l \otimes 1} S/I \otimes_R S.$$

From the exact sequence we derive the following exact sequence:

$$0 \to (s^{-1}I)\, S \to S \xrightarrow{s_l} S/I\, S,$$

because $(S/I) \otimes_R S \cong (S \otimes_R S)/(I \otimes_R S) \cong S/(IS)$ by Lemma 5.1.1, which shows that $(s^{-1}I)S \supseteq s^{-1}(IS)$. Hence $(s^{-1}I)S = s^{-1}(IS)$ follows.

A family $\mathcal{F}$ of right ideals of R is called a *right Gabriel topology* on R if $\mathcal{F}$ satisfies the following two conditions:

(i) if $I \in \mathcal{F}$ and $r \in R$, then $r^{-1}I \in \mathcal{F}$, and

(ii) if $I \in \mathcal{F}$ and J is a right ideal of R such that $a^{-1}J \in \mathcal{F}$ for all $a \in I$, then $J \in \mathcal{F}$.

If $\mathcal{F}$ is a right Gabriel topology on R, then we write $R_{\mathcal{F}}$ for the right quotient of R with respect to $\mathcal{F}$. Since R is a prime Goldie ring, $R_{\mathcal{F}} = \cup\{(R:I)_l \mid I \in \mathcal{F}\}$, where $(R:I)_l = \{q \in Q \mid qI \subseteq R\}$. We refer the readers to [S] for elementary properties of Gabriel topology.

Proposition 5.1.3. *Let S be an overring of R. Suppose that S is flat as a left R-module. Then $\mathcal{F}(S) = \{I$: right ideal of $R \mid IS = S\}$ is a right Gabriel topology on R and $S = R_{\mathcal{F}(S)}$.*

Proof. Let $J \in \mathcal{F}(S)$ and $r \in R$ $(r \neq 0)$. Then $R/(r^{-1}I) \cong (rR + I)/I$ implies $(r^{-1}I)S = S$, i.e., $r^{-1}I \in \mathcal{F}(S)$, because $IS = S$. Next, let $I \in \mathcal{F}(S)$ and let J be a right ideal of R such that $(a^{-1}J)S = S$ for all $a \in I$. Then $S \supseteq JS \supseteq \sum_{a\in I} a(a^{-1}J)S = \sum_{a\in I} aS = S$. Thus $JS = S$, i.e., $J \in \mathcal{F}(S)$. Hence $\mathcal{F}(S)$ is a right Gabriel topology on R.

To show that $S = R_{\mathcal{F}(S)}$, let $I \in \mathcal{F}(S)$. Then $S = RS \supseteq (R:I)_l\, IS \supseteq (R:I)_l$, which implies $R_{\mathcal{F}(S)} \subseteq S$. To show the converse inclusion, let $s \in S$. Then $S = s^{-1}S = (s^{-1}R)S$ by Lemma 5.1.2 and so $s^{-1}R \in \mathcal{F}(S)$ and $s \in (R: s^{-1}R)_l \subseteq R_{\mathcal{F}(S)}$ Hence $S = R_{\mathcal{F}(S)}$.

Corollary 5.1.4. *Under the same notation and assumptions as in Proposition 5.1.3, let I' be a right ideal of S. Then $I' = (I' \cap R)S$.*

Since any overring of a Prüfer ring R is flat as a right R-module as well as a left R-module, we have

Corollary 5.1.5. *Let S be an overring of a Prüfer ring R. Then there is a right (left) Gabriel topology $\mathcal{F}$ ($\mathcal{F}'$) on R such that $S = R_{\mathcal{F}} = R_{\mathcal{F}'}$.*

5.2. Prime ideals of overrings of a PI Prüfer ring.

In this section, we assume that R is a PI Prüfer ring. Note that R_P is a Dubrovin valuation ring for any $P \in \text{Spec}(R)$ and that any overring of a Prüfer ring is Prüfer (see [MMU, (2.6)]).

Lemma 5.2.1. *Let* $P \in \text{Spec}(R)$ *and* $P_1' \in \text{Spec}(R_P)$. *Then* $P_1 = P_1' \cap R \in \text{Spec}(R)$ *and* $R_{P_1} = (R_P)_{P_1'}$.

Proof. Since $J((R_P)_{P_1'}) \cap R_P = P_1'$, we have $P_1 = J((R_P)_{P_1'}) \cap R$ and so $P_1 \in \text{Spec}(R)$ by [Mo, (1.8)]. Since $J(R_P) \supseteq P_1'$, we have $P = J(R_P) \cap R \supseteq P_1$ and so $C(P) \supseteq C(P_1)$ by [MMU, (17.1)]. To prove that $C(P_1) \subseteq C_{R_P}(P_1') = \{\alpha \in R_P \mid \alpha \text{ is regular mod } P_1'\}$, let $c \in C(P_1)$ and $c\beta \in P_1'$ for some $\beta \in R_P$. Then there is a $d \in C(P)$ and $\beta d \in R$, i.e., $c\beta d \in P_1$ and so $\beta d \in P_1$. Thus $\beta \in P_1'$ and hence $C(P_1) \subseteq C_{R_P}(P_1')$. This implies that $R_{P_1} \subseteq (R_P)_{P_1'}$. To prove the converse inclusion, we claim that $\alpha c \in C(P_1)$ for any $\alpha \in C_{R_P}(P_1')$ and $c \in C(P)$ with $\alpha c \in R$. Assume that $\alpha cr \in P_1$ for some $r \in R$. Then $cr \in P_1'$ and so $r \in P_1' \cap R = P_1$, because $C(P) \subseteq C(P_1) \subseteq C_{R_P}(P_1')$. Hence $\alpha c \in C(P_1)$. Now, let $x \in (R_P)_{P_1'}$. Then $x\beta \in R_P$ for some $\beta \in C_{R_P}(P_1')$ and so $x\beta c \in R$ for some $c \in C(P)$ with $\beta c \in R$. Since $\beta c \in C(P_1)$, we have $x \in R_{P_1}$. Hence $R_{P_1} = (R_P)_{P_1'}$ follows.

Lemma 5.2.2. *Let S be an overring of R and let* $P' \in \text{Spec}(S)$. *Then* $P = P' \cap R \in \text{Spec}(R)$ *and* $R_P = S_{P'}$.

Proof. Since $P = P' \cap R = J(S_{P'}) \cap S \cap R = J(S_{P'}) \cap R$, it follows from [Mo, (1.8)] that $P \in \text{Spec}(R)$. Let $c \in C(P)$ and assume that $cs \in P'$ with $s \in S$. Then there is an $I \in \mathcal{F}(S)$ with $xI \subseteq R$ and so $cxI \subseteq P$. Thus $xI \subseteq P$ and so $x \in P'$, which implies $C(P) \subseteq C_S(P')$. Hence $R_P \subseteq S_{P'}$. Since R_P is a Dubrovin valuation ring, there is a $P_1' \in$

Spec(R_P) with $S_{P'} = (R_P)_{P_1'}$ by Theorem 1.2.6. We put $P_1 = J((R_P)_{P_1'}) \cap R$. Then $P_1 = J(S_{P'}) \cap R = P$. Hence $R_P = R_{P_1} = (R_P)_{P_1'} = S_{P'}$ by Lemma 5.2.1.

Theorem 5.2.3. *Let R be a PI Prüfer ring and let S be an overring of R. Then* Spec(S) $= \{PS \mid P \in$ Spec(R) *with* $PS \subset S\}$ *and* $S = \cap R_P$, *where P runs over all* $P \in$ Spec(R) *with* $PS \subset S$.

Proof. Let $P \in$ Spec(R) with $PS \subset S$ and let I' be a maximal right ideal of S with $I' \supseteq PS$. Then $M' = ann_S(S/I') = \{s \in S \mid (S/I')s = 0\}$ is a maximal ideal of S and so $P_0 = M' \cap R \in$ Spec(R) with $R_{P_0} = S_{M'}$ by Lemma 5.2.2. Because of $S/M' \cong S_{M'}/(M'S_{M'}) \cong Q(S/M')$, the quotient ring of S/M', we have $PS_{M'} \subseteq I_{M'} \subset S_{M'}$. So if $P_0 \not\supseteq P$, then $R_{P_0} = PR_{P_0} = PS_{M'} \subset S_{M'}$, a contradiction. Thus $P_0 \supseteq P$ and so $R_P \supseteq R_{P_0} \supseteq S$ by [MMU, (17.1)]. Hence $S \subseteq \cap R_P$, where P runs over all $P \in$ Spec(R) with $PS \subset S$. Furthermore, for any $P \in$ Spec(R) with $PS \subset S$, let $P' = J(R_P) \cap S$. Then $P' \in$ Spec(S) by [Mo, (1.8)]. Since $P = J(R_P) \cap R = P' \cap S$, it follows from Corollary 5.1.4 that $P' = PS$. Hence $PS \in$ Spec(S). Conversely, let $P' \in$ Spec(S) with $S \supset P'$. Then $P = P' \cap R \in$ Spec(R), $P' = PS$ and $R_P = S_{P'}$ by Corollary 5.1.4 and Lemma 5.2.2. Hence $S = \cap R_P$ by [MMU, (14.6)], where P runs over all $P \in$ Spec(R) with $PS \subset S$ and Spec(S) $= \{P \in$ Spec(R) $\mid PS \subset S\}$.

Corollary 5.2.4. *Let R be a PI Prüfer ring and let* $P_i \in$ Spec(R) (i = 1, 2)*. Then* $P_1 + P_2 = R$ *or* $P_1 \supseteq P_2$ *or* $P_1 \subseteq P_2$.

Proof. Suppose that $R \supseteq P_1 + P_2$. Let M be a maximal ideal of R with $M \supseteq P_1 + P_2$. Then $P_i R_M \in$ Spec(R_M) by Theorem 5.2.3 and so either $P_1 R_M \supseteq P_2 R_M$ or $P_1 R_M \subseteq P_2 R_M$ by Proposition 1.2.5. Since $C(M) \subseteq C(P_i)$ by [MMU, (17.1)], we have $P_i R_M \cap R = P_i$. Hence either $P_1 \supseteq P_2$ *or* $P_1 \subseteq P_2$.

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