Proposal:

Considering proof for discovery: teaching materials concerning geometry.

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1. Introduction

In addition to the importance of the curriculum development in teaching of mathematical proof, the development of teaching materials for teaching of mathematical proof is important as well, especially when we work with making up actual classes.

Because, in actual classes, we have to prompt students to make their own mathematical proof to establish some statements of their own accord. In other words, we have to transfer the responsibility for the mathematical statement or mathematical question to the students and its success depends much on the teaching materials.

Thus, first, I'd like to ask: "why do we prove?" What benefit do we obtain from considering proof?

Of course, so many benefits are there! In fact, De Villiers(1990) pointed out that mathematical proof has not only the function of verification but several functions, such as explanation, systematization, discovery and so on. And now, I'd like to give some simple demonstrations of the function of discovery, by considering some problems concerning geometry.

2. Characterization of Parallelogram

Recall the characterizations of a parallelogram. In other words, what condition is sufficient for a quadrangle to be a parallelogram? In Japanese secondary schools, usually, the following five conditions are taken up in classes:

・two pairs of opposite sides are parallel.（Definition）

・two pairs of opposite sides are equal in length.

・two pairs of opposite angles are equal in measure.

・one pair of opposite sides are parallel and equal in length.

・diagonal to other bisects each other.

By the way, in a class of geometry in Japanese pre-service mathematics teacher education, a student muttered the following question: is it true that the next condition is a characterization of parallelogram?

"A pair of opposite sides is equal in length and a pair of opposite angles is equal in measure."(Condition P)

It is clear that all parallelograms satisfy the condition P, but it is subtle whether a

quadrangle satisfying condition P is always a parallelogram or not. Against this question, even though you may think it is doubtful, it seems to be rather hard to hit upon a counter example. On the other hand, what happens when you try to prove that this *is* a characterization? Even if you do *not* think this condition P is a characterization of parallelogram, it is worth trying.

Usually, when we prove that each of the above known characterizing condition actually implies that the given quadrangle is a parallelogram, we use the congruence of two triangles obtained by dividing the quadrangle by one or two diagonals. Now, consider a quadrangle ABCD satisfying the condition P i.e., AB=CD and ∠B=∠D, and try to show that two triangles △ABC and △CDA are congruent (Fig. 1). As can be seen easily, AC is a common side of both triangles, AB=CD and ∠B=∠C. Therefore, among these

triangles, two pairs of sides are equal and one pair of angles is equal in measure, but it become clear that we can't conclude that these triangles are congruent, since each same-measure angle is not between the same-length sides. Indeed, the following fact is written in many textbooks of mathematics for junior-high school in Japan (Fig.2):
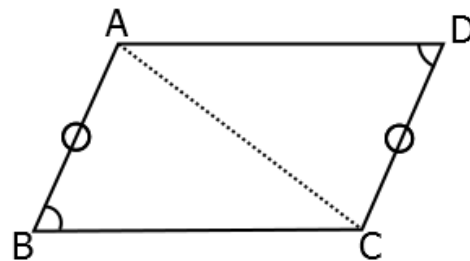

Figure 1. Condition P

上の Q で，右の図のような位置に，2辺と1つの角を決めても，三角形は2通り考えられる。

これに対し，1つの角の位置を「2辺の間」と決めれば，三角形は1通りに決まる。

Figure 2. △ABC and △A"B"C" are not congruent.

Thus, for two triangles △XYZ and △X'Y'Z', even if XY=X'Y', YZ=Y'Z' and ∠X=∠X', △XYZ and △X'Y'Z' may not be congruent. (*)

These facts found in this consideration lead us to the idea how to construct the counter example. That is, to construct two triangles indicating the counter example of (*) and combine them. This goes well as shown in the Fig.3.

At the beginning, the statement "condition P implies that the quadrangle is a parallelogram" was subtle and we could not imagine the counter example even if we believed that is false, however, the thought of trying to prove this statement enlightens us on the mathematical structures of the objects under consideration, and bring us, if anything, to the counter example. Thus trying to prove can induce consideration within the understanding of the structure of objects, and this promote the discovery.
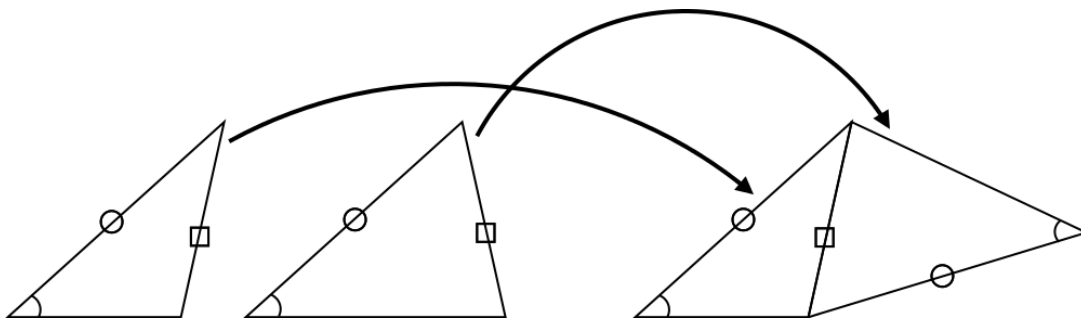

Figure 3. Counter example

3. Generalized golden section

The second demonstration is about the so called golden section. Golden section is the division of a rectangle X into a square Y and a smaller rectangle X', which is similar to the original rectangle X (See Fig.4). One day in a seminar, we considered generalized golden sections, that is, a division of a quadrangle into two parts, one of which is similar to the original quadrangle, using one parting segment. This seminar was held in a context of Japanese pre-service mathematics teacher education. Can you imagine such a generalized golden section, which is different from the classical golden section? Here we consider this question as a teaching material.
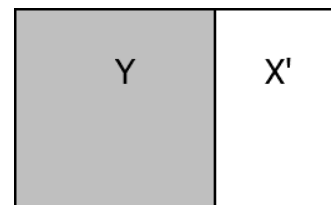

Figure 4. Golden section

Inquiry by the students

Against this problem, my students looked for the examples in a heuristic way. And they found that any rectangle, which is not a square, can be divided as a generalized golden section (Fig.5 left). Also, they found that any parallelogram, which is not a rhombus, can be divided as well (Fig.5 right). However, they couldn't find any more example, and became suspicious of the existence of more example, though they couldn't assert the non-existence clearly: they are stuck.

At this moment, what can lead them further is not a heuristic consideration, but a logical consideration, that is, considering proof. I asked them "*if you do have another*
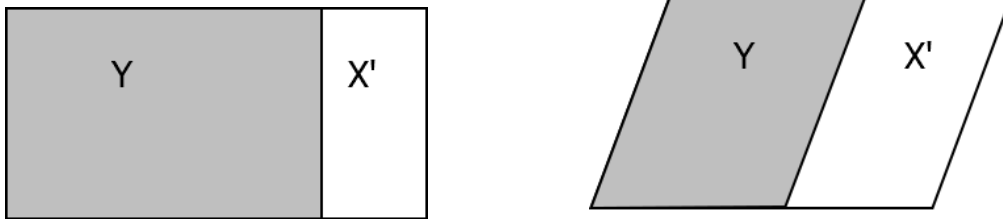
Figure 5. Generalized golden sections

quadrangle which can be divided as a generalized golden section, what can be said about the quadrangle?" The existence of more example is obscure, however, logical thinking allows them to assume its existence, and to look into the property of the assumed quadrangle. In fact, this is nothing but a consideration of the necessary condition of a quadrangle to be divided as a generalized golden section.

At the beginning of this consideration, they hesitated to assume a quadrangle, drawn randomly on the blackboard, to be divided as a generalized golden section, but immediately they found the merit of this consideration.

First, once assumed the existence and drawn on the blackboard, they could notice that there are only two cases: one case where quadrangle is divided into two quadrangles, and the other case where that's divided into a triangle and a quadrangle. Then, next, they can examine more precise properties of the quadrangle for each case. For example, we consider the latter case here. If the whole quadrangle ABCD is divided into a triangle ABE and a similar quadrangle BCDE (Fig.6), we can proceed the study of this quadrangle by distinguishing the correspondence of the vertexes between the similar quadrangles. Some correspondences lead to a contradiction, and others lead to the precise information of the quadrangle. For instance, if A, B, C and D corresponds to B, C, D and E respectively in the similarity, we can say that the three angles ∠B, ∠C, ∠D are equal in measure, and DC:CB=CB:BA. These information lead us to the new example, which could never been found in a heuristic consideration (Fig. 6).
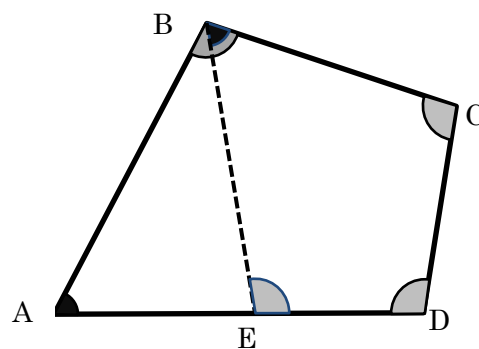

Figure 6. Generalized golden section

Advantage as a teaching material

At a glance, this problem of looking for generalized golden sections seems to require much inspiration, however, logical thinking is rather essential in solving this problem as mentioned above. In fact, one can find many examples by deductive consideration,

which could never found in a heuristic consideration. Thus, by coping with this problem, a learner would find the advantage of considering proof, the function of discovery. In fact, further consideration leads us to the statement which describes the necessary and sufficient condition for a quadrangle to be divided as a generalized golden section (Hamanaka, 2015a, 2015b).

Also, the key in this deductive thinking is the assumption of the existence of what they looking for, which is offered by Teacher. This assumption promote focusing on the necessary condition for the objects to satisfy the property under the consideration. In Japanese senior high school, these concepts about logic, including "necessary condition", "sufficient condition", are studied in the first grade, however, these concepts often lose substances for students and are not accepted with substantial meanings by most students, because of the seldom opportunity to learn these concepts in real proving activities or inquiries. On the other hand, in the inquiries against this golden section's problem, the concept of "necessary condition" rises with its raison d'etre and moreover the explicit efficiency of the method using this concept of "necessary condition" can be demonstrated to the learners.

4. Conclusion

The both of above examples illustrate well one of the value of the thought of proving, power of discovery. To emphasize the value of mathematical proofs in teaching, what is important is not to prove the statement whose genuineness is clear (or whose genuineness is implied by teacher's indication) and not to construct the formal and conventional proof, but to understand the structure under consideration for the examination of some statement and to construct a logical argument for judgement of the statement in a responsible way. In order to do that, these described teaching materials seem valuable. And also, we have to develop more teaching materials, by which we can promote such a consideration.

References

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